Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 53 (2007)
Heft: 3-4

Artikel: The combinatorial cost
Autor: Elek, Gábor
DOI: https://doi.org/10.5169/seals-109545

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 26.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

L'Enseignement Mathématique (2) 53 (2007), 225–235

THE COMBINATORIAL COST

by Gábor ELEK *)

ABSTRACT. We study the combinatorial analogues of the classical invariants of measurable equivalence relations. We introduce the notion of cost and β-invariants (the analogue of the first L^2-Betti number introduced by Gaboriau [3]) for sequences of finite graphs with uniformly bounded vertex degrees and examine the relation of these invariants and the rank gradient resp. mod p homology gradient invariants introduced by Lackenby ([5], [6]) for residually finite groups.

1. INTRODUCTION

1.1 GRAPH SEQUENCES

Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a sequence of finite simple graphs satisfying the following conditions:

- $\sup_{1\leq n<\infty} \max_{x\in V(G_n)} \deg(x) < \infty$. That is, the graphs have uniformly bounded vertex degrees.
- $|V(G_n)| \to \infty$ as $n \to \infty$.

In the sequel we refer to such systems as *graph sequences*. Now let $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ be another graph sequence such that $V(H_n) = V(G_n)$ for any $n \geq 1$. Then $\mathcal{H} \prec \mathcal{G}$ if there exists an integer $L > 0$ such that for any $n \geq 1$ and $x, y \in V(H_n)$, $d_{G_n}(x,y) \leq L\, d_{H_n}(x,y)$, where d_{G_n} resp. d_{H_n} denote the shortest path metrics on G_n resp. on H_n. That is, if x and y are adjacent in the graph H_n then there exists a path between x and y in G_n of length at most L. We say that $\mathcal{G}$ and $\mathcal{H}$ are *equivalent*, $\mathcal{G} \simeq \mathcal{H}$, if $\mathcal{H} \prec \mathcal{G}$ and $\mathcal{G} \prec \mathcal{H}$. The *edge measure* of $\mathcal{G}$ is defined as

$$e(\mathcal{G}) := \liminf_{n\to\infty} \frac{|E(G_n)|}{|V(G_n)|}$$

*) The author is supported by OTKA Grants T 049841 and T 037846.

and the *cost* of $\mathcal{G}$ is given as

$$c(\mathcal{G}) := \inf_{\mathcal{H} \simeq \mathcal{G}} e(\mathcal{H}).$$

Clearly, $c(\mathcal{G}) \geq 1$ for any graph sequence $\mathcal{G}$. Originally, the cost was defined for measurable equivalence relations by Levitt [7]. In our paper we view graph sequences as the analogues of L-graphings of measurable equivalence relations (see [4]).

Recall that a graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is a *large girth sequence* if for any $k \geq 1$, there exists n_k such that if $n \geq n_k$ then G_n does not contain a cycle of length not greater than k. Large girth sequences are the analogues of L-treeings [4]. Our first goal is to prove the following version of Gaboriau's Theorem [2], (see also [4], Theorem 19.2).

THEOREM 1.1. *If* $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ *is a large girth sequence, then* $e(\mathcal{G}) = c(\mathcal{G})$.

1.2 β-INVARIANTS

In the proof of Theorem 1.1 we shall use the β-invariants which are the analogues of the first L^2-Betti numbers of measurable equivalence relations [3]. First recall the notion of cycle spaces.

Let $G(V, E)$ be a finite, simple, connected graph and K be a commutative field. Let $\varepsilon_K(G)$ be the vector space over K spanned by the edges and let $C_K(G) \subseteq \varepsilon_K(G)$, the *cycle space*, be the subspace generated by the cycles of G. Then $\dim_K C_K(G) = |E| - |V| + 1$. Now let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a graph sequence. Let $C_K^q(G_n)$ be the space spanned by the cycles of G_n of length not greater than q. Here we use the usual convention that $(x, y) = -(y, x)$ and we associate to the cycle $(x_1, x_2, \ldots, x_n, x_1)$ the vector $\left(\sum_{i=1}^{n-1}(x_i, x_{i+1}) + (x_n, x_1)\right)$.

Set

$$s_K^q(\mathcal{G}) := \liminf_{n \to \infty} \frac{|E(G_n)| - \dim_K C_K^q(G_n)}{|V(G_n)|} - 1.$$

The β_K*-invariant* of $\mathcal{G}$ is defined as

$$\beta_K(\mathcal{G}) := \inf_q s_K^q(\mathcal{G}).$$

In Section 2 we shall prove that if $\mathcal{G} \simeq \mathcal{H}$, then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$. This immediately shows that

$$\beta_K(\mathcal{G}) + 1 \leq c(\mathcal{G}).$$

1.3 Residually finite groups

Let Γ be a finitely generated group and

$$\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \dots, \quad \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$$

be a nested sequence of finite index normal subgroups. Following Lackenby [5] we define the *rank gradient* of the system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$

$$\text{rk grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \lim_{n\to\infty} \frac{d(\Gamma_n)}{|\Gamma : \Gamma_n|},$$

where $d(\Gamma_n)$ is the minimal number of generators for Γ_n. In another paper [6], Lackenby investigated the behaviour of the sequence $\{\frac{d_p(\Gamma_n)}{|\Gamma : \Gamma_n|}\}_{n=1}^{\infty}$, where $d_p(\Gamma_n) = \dim_{\mathbf{F}_p} H_1(\Gamma_n, \mathbf{F}_p)$. Here we denote by $\mathbf{F}_p$ the finite field of p elements. Note that $d_p(\Gamma_n) \leq d(\Gamma_n)$. The *mod-$p$-homology gradient* of the system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ is defined as

$$p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \liminf_{n\to\infty} \frac{d_p(\Gamma_n)}{|\Gamma : \Gamma_n|}.$$

Let S be a symmetric generating system for Γ and let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be the graph sequence of the Cayley-graphs of Γ/Γ_n with respect to S. We have the following theorem:

THEOREM 1.2. $c(\mathcal{G}) - 1 \leq \text{rk grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$.

If Γ is even finitely presented, then we have the inequality

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \beta^1_{(2)}(\Gamma) \leq p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \beta_{\mathbf{F}_p}(\mathcal{G}) \leq c(\mathcal{G}) - 1,$$

where $\beta^1_{(2)}(\Gamma)$ is the first L^2-Betti number of Γ (see [8]).

1.4 Hyperfinite graph sequences

One of the key notions in the theory of measurable equivalence relations is *hyperfiniteness*. We introduce a similar notion for graph sequences. We shall prove the following analogues of Proposition 22.1 and Lemma 23.2 of [4].

PROPOSITION 1.3.

1. *If $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is a hyperfinite graph sequence then $c(\mathcal{H}) = 1$.*

2. *For any graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ there exists a hyperfinite graph sequence $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ such that $\mathcal{H} \prec \mathcal{G}$.*

Finally we prove the analogue of the theorem of Connes, Feldman and Weiss ([4], Theorem 10.1).

THEOREM 1.4. *Let Γ be a finitely generated residually finite group with a nested sequence of finite index normal subgroups Γ_n, $\bigcap_{n=1}^{\infty}\Gamma_n = \{1\}$. Then the associated graph sequence $\mathcal{G}$ is hyperfinite if and only if Γ is amenable.*

2. β-INVARIANTS

PROPOSITION 2.1. *Let $\mathcal{G} \simeq \mathcal{H}$ be equivalent graph sequences and K be a field. Then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$.*

Proof. Suppose that $\mathcal{H} \subseteq \mathcal{G}$, that is for any $n \geq 1$, $E(H_n) \subseteq E(G_n)$. Let $L > 0$ be an integer such that $d_{G_n}(x,y) \leq L\, d_{H_n}(x,y)$. We define a K-linear transformation between quotient spaces:

$$\widetilde{\phi}\colon \varepsilon_K(H_n)/C_K^q(H_n) \to \varepsilon_K(G_n)/C_K^q(G_n)$$

by extending the inclusion $\phi\colon E(H_n) \to E(G_n)$.

LEMMA 2.2. *If $\widetilde{\phi}$ is surjective then $q > L$.*

Proof. Let $e = (x,y) \in E(G_n)$, then there exists a path P between x and y, in H_n of length not greater than L. The cycle $c = P \cup e$ represents an element in $C_K^q(G_n)$ and

$$[e] \in [c] \oplus [\widetilde{\phi}(\varepsilon_K(H_n))]\,.$$

Hence the lemma follows. $\square$

By the lemma it follows that $s_K^q(H_n) \geq s_K^q(G_n)$ if $q > L$, thus $\beta_K(\mathcal{H}) \geq \beta_K(\mathcal{G})$.

Now we define another K-linear transformation:

$$\widetilde{\psi}\colon \varepsilon_K(G_n)/C_K^q(G_n) \to \varepsilon_K(H_n)/C_K^{qL}(H_n)\,,$$

by mapping the basis vector $e = (x,y) \in E(G_n)$ to a path in H_n of length not greater than L connecting x and y. If $e \in H_n$, then let $\widetilde{\psi}(e) = e$. Obviously, $\widetilde{\psi}$ is surjective therefore $s_K^q(G_n) \geq s_K^{qL}(H_n)$ and consequently $\beta_K(\mathcal{G}) \geq \beta_K(\mathcal{H})$.

Hence if $\mathcal{G} \simeq \mathcal{H}$, $\mathcal{H} \subseteq \mathcal{G}$ then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$. Now we consider the general case, where $\mathcal{H}$ is an arbitrary graph sequence such that $\mathcal{H} \simeq \mathcal{G}$. Then let $\mathcal{J} = \mathcal{G} \cup \mathcal{H}$, that is $V(J_n) = V(G_n), E(J_n) = E(G_n) \cup E(H_n)$. Clearly, $\mathcal{J} \simeq \mathcal{G} \simeq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{J}$, $\mathcal{G} \subseteq \mathcal{J}$. Thus by our argument above, $\beta_K(\mathcal{H}) = \beta_K(\mathcal{J}) = \beta_K(\mathcal{G})$. $\square$

PROPOSITION 2.3. *Let* $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ *be a graph sequence. Then*

$$\beta_{\mathbf{Q}}(\mathcal{G}) \leq \beta_{\mathbf{F}_p}(\mathcal{G}) \leq c(\mathcal{G}) - 1 .$$

Proof. Let $\mathcal{H} \simeq \mathcal{G}$, then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H}) \leq e(\mathcal{H}) - 1$. Therefore $\beta_K(\mathcal{G}) \leq c(\mathcal{G}) - 1$.

LEMMA 2.4. $\dim_{\mathbf{Q}} C^q_{\mathbf{Q}}(G_n) \leq \dim_{\mathbf{F}_p} C^q_{\mathbf{F}_p}(G_n)$.

Proof. Let c^q_n be the number of cycles in G_n of length not greater than q. Let $\rho_{\mathbf{Z}}\colon \mathbf{Z}^{c^q_n} \to \mathbf{Z}^{|E(G_n)|}$ be the homomorphism that maps $\bigoplus_{i=1}^{c^q_n} s_i$ to $\sum_{i=1}^{c^q_n} s_i[c_i]$, where $s_i \in \mathbf{Z}$ and $[c_i]$ is the integer vector generated by the i-th cycle c_i. Similarly, we define $\rho_{\mathbf{F}_p}\colon \mathbf{F}_p^{c^q_n} \to \mathbf{F}_p^{|E(G_n)|}$. Let $\pi_1\colon \mathbf{Z}^{c^q_n} \to \mathbf{F}_p^{c^q_n}$, $\pi_2\colon \mathbf{Z}^{|E(G_n)|} \to \mathbf{F}_p^{|E(G_n)|}$ be the residue class maps. Then $\pi_2 \circ \rho_{\mathbf{Z}} = \rho_{\mathbf{F}_p} \circ \pi_1$. Therefore,

$$\operatorname{rank}_{\mathbf{Z}} \operatorname{Im} \rho_{\mathbf{Z}} \geq \dim_{\mathbf{F}_p} \operatorname{Im} \rho_{\mathbf{F}_p} .$$

Clearly, $\operatorname{rank}_{\mathbf{Z}} \operatorname{Im} \rho_{\mathbf{Z}} = \dim_{\mathbf{Q}} C^q_{\mathbf{Q}}(G_n)$ and $\dim_{\mathbf{F}_p} \operatorname{Im} \rho_{\mathbf{F}_p} = \dim_{\mathbf{F}_p} C^q_{\mathbf{F}_p}(G_n)$. Thus our lemma follows. □

By our lemma, $\beta_{\mathbf{Q}}(\mathcal{G}) \leq \beta_{\mathbf{F}_p}(\mathcal{G})$ hence we finish the proof of our proposition. □

QUESTION 2.5. *Does there exist a graph sequence* $\mathcal{G}$ *for which* $\beta_{\mathbf{Q}}(\mathcal{G}) \neq \beta_{\mathbf{F}_p}(\mathcal{G})$ *or* $\beta_{\mathbf{F}_p}(\mathcal{G}) \neq c(\mathcal{G}) - 1$ *?*

Finally we prove Theorem 1.1.

Proof. Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a large girth graph sequence. Then by definition $\beta_K(\mathcal{G}) = e(\mathcal{G}) - 1$. That is, $e(\mathcal{G}) - 1 \leq c(\mathcal{G}) - 1$, hence our theorem follows. □

3. RESIDUALLY FINITE GROUPS

The goal of this section is to prove Theorem 1.2. Let Γ be a finitely generated residually finite group with a not necessarily symmetric generating system S. Let $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \ldots$, $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$ be a nested sequence of finite index normal subgroups and $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be the graph sequence, where G_n is the (left) Cayley-graph of the finite group Γ/Γ_n with respect

to S. Note that if S' is another generating system and $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is the associated graph sequence then $\mathcal{H} \simeq \mathcal{G}$.

PROPOSITION 3.1. $c(\mathcal{G}) - 1 \leq \operatorname{rk} \operatorname{grad}\left\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\right\}$.

Proof. First note that by the Reidemeister-Schreier theorem the groups Γ_n are finitely generated as well [9], moreover if T is a finite generating system of Γ_n, then

$$d_{G_T^{\Gamma_n}}(x,y) \leq L\, d_{G_S^{\Gamma}}(x,y)$$

for any $x, y \in \Gamma_n$, where G_S^{Γ} resp. $G_T^{\Gamma_n}$ are the Cayley-graphs with respect to S resp. to T, and the Lipschitz constant L depends only on S and T.

LEMMA 3.2. *For any* $k \geq 1$,

$$\frac{d(\Gamma_k)}{|\Gamma : \Gamma_k|} + 1 \geq c(\Gamma).$$

Proof. We use an idea resembling an argument in the proof of Theorem 21.1 of [4]. Let T be a generating system of Γ_k of minimal number of generators. For simplicity we suppose that $T \subset S$. Consider the following graph sequence: $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$, $V(H_n) = \Gamma/\Gamma_n$. If $n \leq k$, let $H_n = G_n$. Set $S_n = \Gamma_k/\Gamma_n$ and let H_n' be the Cayley-graph of S_n with respect to T. Now enumerate the vertices of $V(H_n)\backslash S_n$, $\{x_1, x_2 \ldots, x_{r_n}\}$. For each x_i consider the set of shortest paths in G_n from x_i to the set S_n. Pick the minimal path with respect to the lexicographic ordering. The edges of H_n shall consist of H_n' and the edges of the minimal paths. Define a map $\pi\colon V(H_n) \to S_n$ in the following way. For each $x_i \in V(H_n)\backslash S_n$ let $\pi(x_i) \in S_n$ be the endpoint of the minimal path from x_i to S_n and let $\pi(x) = x$ if $x \in S_n$. By the lexicographic minimality, the union of the paths form a subforest in G_n having exactly $|V(H_n)\backslash S_n|$ edges.

We claim that $\mathcal{H} \simeq \mathcal{G}$. Since $\mathcal{H} \subset \mathcal{G}$, we only need to prove that $\mathcal{G} \prec \mathcal{H}$. Let $n > k$, $x, y \in V(G_n)$. Consider the shortest G_n-path from x to y, $\{x_0, x_1, \ldots x_l\}$, $x_0 = x$, $x_l = y$. Let us consider the sequence of vertices $\{\pi(x_0), \pi(x_1), \ldots \pi(x_l)\}$.

Let $y_1, y_2, \ldots, y_{|\Gamma:\Gamma_k|}$ be a set of coset-representatives with respect to Γ_k. Let t be the maximal word-length of the representatives with respect to S. Then $d_{G_n}(\pi(x), x) \leq t$ for any $x \in V(G_n)$. Therefore, $d_{G_n}(\pi(x_i), \pi(x_{i+1})) \leq 2t + 1$. That is, $d_{H_n}(\pi(x_i), \pi(x_{i+1})) \leq L(2t + 1)$, where L is the Lipschitz-constant defined before the statement of our lemma. Consequently,

$$d_{H_n}(x,y) \leq L\,(2t + 1)\, d_{G_n}(x,y)$$

and therefore $\mathcal{H} \simeq \mathcal{G}$.

For the edge measure of $\mathcal{H}$ we have

$$e(\mathcal{H}) = \liminf_{n\to\infty} \frac{|\Gamma : \Gamma_n| - |\Gamma_k : \Gamma_n| + |E(H'_n)|}{|\Gamma : \Gamma_n|}.$$

The vertex degrees of H'_n are not greater than $2|T| = 2d(\Gamma_k)$, also $|S_n| = |\Gamma_k : \Gamma_k|$. Thus

$$c(\mathcal{G}) \le e(\mathcal{H}) \le \frac{d(\Gamma_k)}{|\Gamma : \Gamma_k|} + 1.$$

Hence the lemma follows. □

Proposition 3.1 is a straightforward consequence of Lemma 3.2. □

Let $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}, S, \mathcal{G}$ be as above. Moreover suppose that Γ is finitely presented. This means that if $\Theta\colon \mathcal{F}_S \to \Gamma$ is the natural map from the free group generated by S to Γ then $\ker\Theta$ is generated by the relations $\{R_1, R_2, \dots, R_l\}$ as a normal subgroup, that is, if $\Theta(\underline{w}) = 1$ then

$$\underline{w} = \prod_{j=1}^{r_{\underline{w}}} \gamma_j R_{i_j} \gamma_j^{-1}, \quad \gamma_j \in \mathcal{F}_S.$$

Let $\widetilde{\Sigma}$ be the usual covering CW-complex constructed from $\{R_i\}_{i=1}^{l}$, the 1-skeleton of $\widetilde{\Sigma}$ is the Cayley-graph of Γ and for each $\gamma \in \Gamma$ and $1 \le i \le l$, we add a 2-cell $\sigma_{\gamma,i}$ such that

$$\partial\sigma_{\gamma,i} = \sum_{j=1}^{s_i} (\underline{w}_j\gamma, \underline{w}_{j-1}\gamma),$$

where $R_i = a_{s_i}a_{s_i-1}\dots a_2a_1$, $\underline{w}_j = a_ja_{j-1}\dots a_2a_1$, $\underline{w}_0 = 1$. Then $\widetilde{\Sigma}$ is simply connected with a natural Γ-action. Clearly, $\pi_1(\widetilde{\Sigma}/\Gamma_n) = \Gamma_n$. Recall that the group homology space $H_1(\Gamma_n, K)$ is isomorphic to the CW-homology space $H_1(\widetilde{\Sigma}/\Gamma_n, K)$.

LEMMA 3.3. *We have*

$$\lim_{n\to\infty} \frac{\dim_K H_1(\widetilde{\Sigma}/\Gamma_n, K)}{|\Gamma : \Gamma_n|} = \beta_K(\mathcal{G}).$$

Proof. Consider the homology complex

$$C_2(\widetilde{\Sigma}/\Gamma_n, K) \xrightarrow{\partial_2} C_1(\widetilde{\Sigma}/\Gamma_n, K) \xrightarrow{\partial_1} C_0(\widetilde{\Sigma}/\Gamma_n, K).$$

Observe that

$$C_1(\widetilde{\Sigma}/\Gamma_n, K) \simeq \varepsilon_K(G_n) \quad \text{and} \quad \dim_K C_0(\widetilde{\Sigma}/\Gamma_n, K) = |V(G_n)|\,.$$

Let r be the maximal word-length of a relation R_i. Then $\partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ is generated by cycles of length at most r. On the other hand, for any $q > r$, the q-cycles are in $\partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ if n is large enough.

Therefore $C_K^q(G_n) = \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ if n is large enough. Consequently,

$$s_K^q(\mathcal{G}) = \liminf_{n\to\infty} \frac{|E(G_n)| - \dim_K \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K)) - |V(G_n)|}{|\Gamma : \Gamma_n|}\,.$$

On the other hand,

$$\begin{aligned}\frac{\dim_K H_1(\widetilde{\Sigma}/\Gamma_n, K)}{|\Gamma : \Gamma_n|} &= \frac{\dim_K \ker \partial_1 - \dim_K \operatorname{Im} \partial_2}{|\Gamma : \Gamma_n|} \\ &= \frac{|E(G_n)| - \dim_K \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K)) - |V(G_n)| + 1}{|\Gamma : \Gamma_n|}\,.\end{aligned}$$

Hence the lemma follows. □

Now we prove the second part of Theorem 1.2.

PROPOSITION 3.4. *Let Γ be a finitely presented residually finite group, $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}, S, \mathcal{G}$ be as above. Then*

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \beta_{(2)}^1(\Gamma) \le p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \beta_{\mathbf{F}_p}(\mathcal{G}) \le c(\mathcal{G}) - 1\,,$$

where $\beta_{(2)}^1(\Gamma)$ is the first L^2-Betti number of Γ (see [8]).

Proof. By Lemma 3.3, $\beta_{\mathbf{F}_p}(\mathcal{G}) = p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$. Also,

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \liminf_{n\to\infty} \frac{\dim_{\mathbf{Q}} H_1(\widetilde{\Sigma}/\Gamma_n, \mathbf{Q})}{|\Gamma : \Gamma_n|}\,.$$

By the Approximation Theorem of Lück

$$\lim_{n\to\infty} \frac{\dim_{\mathbf{Q}} H_1(\widetilde{\Sigma}/\Gamma_n, \mathbf{Q})}{|\Gamma : \Gamma_n|} = \beta_{(2)}^1(\Gamma)\,.$$

Hence our proposition follows. □

QUESTION 3.5. 1. *Does there exist a finitely presented residually finite group Γ and a system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ such that*

$$\beta_{(2)}^1(\Gamma) \ne p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} \quad \textit{or} \quad p\text{-grad}\,\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} \ne c(\mathcal{G}) - 1\,?$$

2. *Does there exist a finitely generated residually finite group* Γ *and a system* $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ *such that*

$$c(\mathcal{G}) - 1 \neq \text{rk grad}\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}\,?$$

4. Hyperfinite graph sequences

We say that a graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is *hyperfinite* if for any $\epsilon > 0$ there exists $K_\epsilon > 0$, positive integers $\{k_n\}_{n=1}^{\infty}$ and a sequence of partitions of the vertex sets $V(G_n)$

$$A_1^n \cup A_2^n \cup \cdots \cup A_{k_n}^n = V(G_n)$$

such that

- For any $n \geq 1$, $1 \leq i \leq k_n$, $|A_i^n| \leq K_\epsilon$.
- If E_n^ϵ is the set of edges $(x, y) \in E(G_n)$ such that $x \in A_i$, $y \in A_j$, $x \neq y$, then

$$\liminf_{n\to\infty} \frac{|E_n^\epsilon|}{|V(G_n)|} \leq \epsilon\,.$$

Now we prove Proposition 1.3.

Proof. Suppose that $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is hyperfinite. Let $\mathcal{H}^\epsilon = \{H_n^\epsilon\}_{n=1}^{\infty}$ be the following graph sequence. The vertex set of H_n^ϵ is $V(G_n)$, $E(H_n^\epsilon)$ is the union of E_n^ϵ and a spanning tree for each connected component of the graphs spanned by the vertices of A_i^n, $1 \leq i \leq k_n$. Clearly, $\mathcal{H}^\epsilon \simeq \mathcal{G}$ and $|E(H_n^\epsilon)| \leq |E_n^\epsilon| + |V(G_n)|$ thus $e(\mathcal{H}^\epsilon) \leq 1 + \epsilon$. Therefore $c(\mathcal{G}) = 1$.

Now we show that for any graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$, $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is hyperfinite where H_n is a spanning tree of G_n. We actually show that a sequence of trees $\mathcal{T} = \{T_n\}_{n=1}^{\infty}$ is always hyperfinite. Let q be an integer and consider a maximal q-net $L_n^q \subset V(T_n)$. That is, if $x \neq y \in L_n^q$ then $d_{T_n}(x, y) \geq q$ and for any $z \in V(T_n)$ there exists $x \in L_n^q$ such that $d_{T_n}(x, z) \leq q$. Now for each $x \in V(T_n)$ let $\pi(x)$ be one of the vertices $y \in L_n^q$ closest to x. Then $\bigcup_{y \in L_n^q} \pi^{-1}(y)$ is a partition of $V(T_n)$. Clearly $|\pi^{-1}(y)| \geq q$ for any $y \in L_n^q$. Obviously the T_n^y subgraph spanned by the vertices in $\pi^{-1}(y)$ is connected. Thus

$$|E_n^\epsilon| \leq |V(T_n)| - (|V(T_n)| - |L_n^q|)\,.$$

Here we used the fact that a connected graph has at least as many edges as the number of its vertices minus one. Obviously, $|L_n^q| \leq \frac{|V(T_n)|}{q}$, therefore

$$\lim_{n\to\infty} \frac{|E_n^c|}{|V(T_n)|} \leq \frac{1}{q}.$$

Consequently, the graph sequence $\mathcal{T}$ is indeed hyperfinite. □

Finally, we prove Theorem 1.4.

Proof. First let Γ be a residually finite non-amenable group with a symmetric generating system S and a nested sequence of finite index normal subgroups $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \ldots$, $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. Let G_n be the Cayley-graph of Γ/Γ_n with respect to S and G_S^Γ be the Cayley-graph of the group Γ. Since Γ is non-amenable, it has no Følner-exhaustion, consequently there exists a real number $\delta > 0$ such that for each finite subset $F \subset \Gamma$ the number of edges from F to the complement of F is at least $\delta|F|$. Fix an integer $m > 0$. If n is large enough then for any subset $M \subset \Gamma/\Gamma_n$, $|M| \leq m$ the number of edges from M to its complement must be at least $\delta|M|$. This follows easily form the fact that for any $r \geq 0$, the r-balls in G_n and in G_S^Γ are isometric. This implies that $\mathcal{G}$ is not hyperfinite.

Now let $\Gamma, \{\Gamma_n\}_{n=1}^{\infty}, S, \mathcal{G}$ be as above, but let Γ be amenable. The following lemma is a straightforward consequence of Theorem 2 of [1].

LEMMA 4.1. *For any $\omega > 0$, there exist $L_\omega > 0$, $M_\omega > 0$ and a sequence of family of subsets*

$$\{W_n^i\}_{i=1}^{k_n}, \quad W_n^i \subset V(G_n) \quad \text{if} \quad n \geq M_\omega$$

such that for any $1 \leq i \leq k_n$,

- $|W_n^i| \leq L_\omega$,
- $|W_n^i \setminus \bigcup_{j\neq i}^{k_n} W_n^j| \geq (1-\omega)|W_n^i|$,
- *the number of edges from W_n^i to its complement is at most $\omega|W_n^i|$,*

and

- $|\bigcup_{i=1}^{k_n} W_n^i| \geq (1-\omega)|V(G_n)|$.

Now let $Z_n^i = W_n^i \setminus \bigcup_{j\neq i}^{k_n} W_n^j$ and consider the partition of $V(G_n)$,

$$V(G_n) = \bigcup_{i=1}^{k_n} Z_n^i \cup \bigcup_{j=1}^{l_n} T_n^i,$$

where T_n^i are arbitrary subsets of size at most L_ω. Let E_n^ω be the set of edges $(x,y) \in G_n$ such that their endpoints belong to different subsets in the partition. There are three kinds of edges in E_n^ω :

- Edges with an endpoint in T_n^i. The number of such edges is at most $2|S|(1-(1-\omega)^2)|V(G_n)|$.
- Edges from Z_n^i to the complement of W_n^i, for some $1 \leq i \leq k_n$. The number of such edges is at most $2|S|\omega(1-\omega)^{-1}|V(G_n)|$.
- Edges from Z_n^i to $W_n^i \backslash Z_n^i$ for some $1 \leq i \leq k_n$. The number of such edges is at most $2|S|\omega(1-\omega)^{-1}|V(G_n)|$.

Hence

$$\liminf_{n\to\infty} \frac{|E_n^\omega|}{|V(G_n)|} \leq 2|S|\left((1-(1-\omega)^2) + 2\omega(1-\omega)^{-1}\right).$$

Therefore $\mathcal{G}$ is hyperfinite. □

REFERENCES

[1] ELEK, G. The strong approximation conjecture holds for amenable groups. *J. Funct. Anal. 239* (2006), 345–355.

[2] GABORIAU, D. Coût des relations d'équivalence et des groupes. *Invent. Math. 139* (2000), 41–98.

[3] —— Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci. 95* (2002), 93–150.

[4] KECHRIS, A. S. and B. D. MILLER. *Topics in Orbit Equivalence Theory*. Lecture Notes in Mathematics *1852*. Springer-Verlag, Berlin, 2004.

[5] LACKENBY, M. Expanders, rank and graphs of groups. *Israel J. Math. 146* (2005), 357–370.

[6] —— Large groups, property (τ) and the homology growth of subgroups. (Preprint).

[7] LEVITT, G. On the cost of generating an equivalence relation. *Ergodic Theory Dynam. Systems 15* (1995), 1173–1181.

[8] LÜCK, W. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, *44*. Springer-Verlag, Berlin, 2002.

[9] MAGNUS, W., A. KARRASS and D. SOLITAR. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*. Interscience Publishers, John Wiley & Sons, Inc., New York-London-Sydney, 1966.

(Reçu le 4 septembre 2006)

Gábor Elek

The Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P.O.B. 127
H-1364 Budapest
Hungary
e-mail: elek@renyi.hu