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THE COMBINATORIAL COST

by Gábor ELEK*)

ABSTRACT. We study the combinatorial analogues of the classical invariants of measurable equivalence relations. We introduce the notion of cost and β -invariants (the analogue of the first L^2 -Betti number introduced by Gaboriau [3]) for sequences of finite graphs with uniformly bounded vertex degrees and examine the relation of these invariants and the rank gradient resp. mod p homology gradient invariants introduced by Lackenby ([5], [6]) for residually finite groups.

1. INTRODUCTION

1.1 GRAPH SEQUENCES

Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a sequence of finite simple graphs satisfying the following conditions:

- $\sup_{1 \le n < \infty} \max_{x \in V(G_n)} \deg(x) < \infty$. That is, the graphs have uniformly bounded vertex degrees.
- $|V(G_n)| \to \infty$ as $n \to \infty$.

In the sequel we refer to such systems as graph sequences. Now let $\mathcal{H}=\{H_n\}_{n=1}^\infty$ be another graph sequence such that $V(H_n)=V(G_n)$ for any $n\geq 1$. Then $\mathcal{H}\prec\mathcal{G}$ if there exists an integer L>0 such that for any $n\geq 1$ and $x,y\in V(H_n),\ d_{G_n}(x,y)\leq L\,d_{H_n}(x,y),$ where d_{G_n} resp. d_{H_n} denote the shortest path metrics on G_n resp. on H_n . That is, if x and y are adjacent in the graph H_n then there exists a path between x and y in G_n of length at most L. We say that \mathcal{G} and \mathcal{H} are equivalent, $\mathcal{G}\simeq\mathcal{H}$, if $\mathcal{H}\prec\mathcal{G}$ and $\mathcal{G}\prec\mathcal{H}$. The edge measure of \mathcal{G} is defined as

$$e(\mathcal{G}) := \liminf_{n \to \infty} \frac{|E(G_n)|}{|V(G_n)|}$$

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and the cost of G is given as

$$c(\mathcal{G}) := \inf_{\mathcal{H} \simeq \mathcal{G}} e(\mathcal{H}).$$

Clearly, $c(\mathcal{G}) \geq 1$ for any graph sequence \mathcal{G} . Originally, the cost was defined for measurable equivalence relations by Levitt [7]. In our paper we view graph sequences as the analogues of L-graphings of measurable equivalence relations (see [4]).

Recall that a graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is a large girth sequence if for any $k \geq 1$, there exists n_k such that if $n \geq n_k$ then G_n does not contain a cycle of length not greater than k. Large girth sequences are the analogues of L-treeings [4]. Our first goal is to prove the following version of Gaboriau's Theorem [2], (see also [4], Theorem 19.2).

THEOREM 1.1. If $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is a large girth sequence, then $e(\mathcal{G}) = c(\mathcal{G})$.

1.2 β -INVARIANTS

In the proof of Theorem 1.1 we shall use the β -invariants which are the analogues of the first L^2 -Betti numbers of measurable equivalence relations [3]. First recall the notion of cycle spaces.

Let G(V,E) be a finite, simple, connected graph and K be a commutative field. Let $\varepsilon_K(G)$ be the vector space over K spanned by the edges and let $C_K(G) \subseteq \varepsilon_K(G)$, the *cycle space*, be the subspace generated by the cycles of G. Then $\dim_K C_K(G) = |E| - |V| + 1$. Now let $G = \{G_n\}_{n=1}^{\infty}$ be a graph sequence. Let $C_K^q(G_n)$ be the space spanned by the cycles of G_n of length not greater than G. Here we use the usual convention that G(x,y) = -(y,x) and we associate to the cycle $G(x_1,x_2,\ldots,x_n,x_n)$ the vector $G(x_1,x_2,\ldots,x_n,x_n)$.

Set

$$s_K^q(\mathcal{G}) := \liminf_{n \to \infty} \frac{|E(G_n)| - \dim_K C_K^q(G_n)}{|V(G_n)|} - 1.$$

The β_K -invariant of \mathcal{G} is defined as

$$\beta_K(\mathcal{G}) := \inf_q s_K^q(\mathcal{G}).$$

In Section 2 we shall prove that if $\mathcal{G} \simeq \mathcal{H}$, then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$. This immediately shows that

$$\beta_{K}(\mathcal{G}) + 1 \leq c(\mathcal{G}).$$

1.3 RESIDUALLY FINITE GROUPS

Let Γ be a finitely generated group and

$$\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \ldots, \quad \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$$

be a nested sequence of finite index normal subgroups. Following Lackenby [5] we define the *rank gradient* of the system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$

rk grad
$$\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = \lim_{n \to \infty} \frac{d(\Gamma_n)}{|\Gamma : \Gamma_n|},$$

where $d(\Gamma_n)$ is the minimal number of generators for Γ_n . In another paper [6], Lackenby investigated the behaviour of the sequence $\left\{\frac{d_p(\Gamma_n)}{|\Gamma:\Gamma_n|}\right\}_{n=1}^{\infty}$, where $d_p(\Gamma_n)=\dim_{\mathbf{F}_p}H_1(\Gamma_n,\mathbf{F}_p)$. Here we denote by \mathbf{F}_p the finite field of p elements. Note that $d_p(\Gamma_n)\leq d(\Gamma_n)$. The mod-p-homology gradient of the system $\left\{\Gamma,\left\{\Gamma_n\right\}_{n=1}^{\infty}\right\}$ is defined as

$$p\operatorname{-grad}\left\{\Gamma, \left\{\Gamma_n\right\}_{n=1}^{\infty}\right\} = \liminf_{n \to \infty} \frac{d_p(\Gamma_n)}{|\Gamma:\Gamma_n|}.$$

Let S be a symmetric generating system for Γ and let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be the graph sequence of the Cayley-graphs of Γ/Γ_n with respect to S. We have the following theorem:

THEOREM 1.2.
$$c(\mathcal{G}) - 1 \le \text{rk grad} \left\{ \Gamma, \left\{ \Gamma_n \right\}_{n=1}^{\infty} \right\}$$
.

If Γ is even finitely presented, then we have the inequality

$$\beta_{\mathbf{O}}(\mathcal{G}) = \beta_{\mathcal{O}}^{\mathbf{I}}(\Gamma) \le p \operatorname{-grad}\left\{\Gamma, \left\{\Gamma_{n}\right\}_{n=1}^{\infty}\right\} = \beta_{\mathbf{F}_{n}}(\mathcal{G}) \le c(\mathcal{G}) - 1,$$

where $\beta_{(2)}^1(\Gamma)$ is the first L^2 -Betti number of Γ (see [8]).

1.4 HYPERFINITE GRAPH SEQUENCES

One of the key notions in the theory of measurable equivalence relations is *hyperfiniteness*. We introduce a similar notion for graph sequences. We shall prove the following analogues of Proposition 22.1 and Lemma 23.2 of [4].

Proposition 1.3.

- 1. If $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is a hyperfinite graph sequence then $c(\mathcal{H}) = 1$.
- 2. For any graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ there exists a hyperfinite graph sequence $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ such that $\mathcal{H} \prec \mathcal{G}$.

Finally we prove the analogue of the theorem of Connes, Feldman and Weiss ([4], Theorem 10.1).

THEOREM 1.4. Let Γ be a finitely generated residually finite group with a nested sequence of finite index normal subgroups Γ_n , $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. Then the associated graph sequence \mathcal{G} is hyperfinite if and only if Γ is amenable.

2. β -INVARIANTS

PROPOSITION 2.1. Let $\mathcal{G} \simeq \mathcal{H}$ be equivalent graph sequences and K be a field. Then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$.

Proof. Suppose that $\mathcal{H} \subseteq \mathcal{G}$, that is for any $n \geq 1$, $E(H_n) \subseteq E(G_n)$. Let L > 0 be an integer such that $d_{G_n}(x,y) \leq L \, d_{H_n}(x,y)$. We define a K-linear transformation between quotient spaces:

$$\widetilde{\phi}$$
: $\varepsilon_K(H_n)/C_K^q(H_n) \to \varepsilon_K(G_n)/C_K^q(G_n)$

by extending the inclusion $\phi: E(H_n) \to E(G_n)$.

LEMMA 2.2. If $\widetilde{\phi}$ is surjective then q > L.

Proof. Let $e=(x,y)\in E(G_n)$, then there exists a path P between x and y, in H_n of length not greater than L. The cycle $c=P\cup e$ represents an element in $C_K^q(G_n)$ and

$$[e] \in [c] \oplus [\widetilde{\phi}(\varepsilon_K(H_n))].$$

Hence the lemma follows. \Box

By the lemma it follows that $s_K^q(H_n) \geq s_K^q(G_n)$ if q > L, thus $\beta_K(\mathcal{H}) \geq \beta_K(\mathcal{G})$.

Now we define another K-linear transformation:

$$\widetilde{\psi}$$
: $\varepsilon_K(G_n)/C_K^q(G_n) \to \varepsilon_K(H_n)/C_K^{qL}(H_n)$,

by mapping the basis vector $e = (x, y) \in E(G_n)$ to a path in H_n of length not greater than L connecting x and y. If $e \in H_n$, then let $\widetilde{\psi}(e) = e$. Obviously, $\widetilde{\psi}$ is surjective therefore $s_K^q(G_n) \geq s_K^{qL}(H_n)$ and consequently $\beta_K(\mathcal{G}) \geq \beta_K(\mathcal{H})$.

Hence if $\mathcal{G} \simeq \mathcal{H}$, $\mathcal{H} \subseteq \mathcal{G}$ then $\beta_K(\mathcal{G}) = \beta_K(\mathcal{H})$. Now we consider the general case, where \mathcal{H} is an arbitrary graph sequence such that $\mathcal{H} \simeq \mathcal{G}$. Then let $\mathcal{J} = \mathcal{G} \cup \mathcal{H}$, that is $V(J_n) = V(G_n), E(J_n) = E(G_n) \cup E(H_n)$. Clearly, $\mathcal{J} \simeq \mathcal{G} \simeq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{J}$, $\mathcal{G} \subseteq \mathcal{J}$. Thus by our argument above, $\beta_K(\mathcal{H}) = \beta_K(\mathcal{J}) = \beta_K(\mathcal{G})$. \square

PROPOSITION 2.3. Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a graph sequence. Then $\beta_{\mathbf{Q}}(\mathcal{G}) \leq \beta_{\mathbf{F}_n}(\mathcal{G}) \leq c(\mathcal{G}) - 1$.

Proof. Let $\mathcal{H}\simeq\mathcal{G}$, then $\beta_K(\mathcal{G})=\beta_K(\mathcal{H})\leq e(\mathcal{H})-1$. Therefore $\beta_K(\mathcal{G})\leq c(\mathcal{G})-1$.

LEMMA 2.4. $\dim_{\mathbb{Q}} C_{\mathbb{Q}}^q(G_n) \leq \dim_{\mathbb{F}_q} C_{\mathbb{F}_q}^q(G_n)$.

Proof. Let c_n^q be the number of cycles in G_n of length not greater than q. Let $\rho_{\mathbf{Z}} \colon \mathbf{Z}^{c_n^q} \to \mathbf{Z}^{|E(G_n)|}$ be the homomorphism that maps $\bigoplus_{i=1}^{c_n^q} s_i$ to $\sum_{i=1}^{c_n^q} s_i[c_i]$, where $s_i \in \mathbf{Z}$ and $[c_i]$ is the integer vector generated by the i-th cycle c_i . Similarly, we define $\rho_{\mathbf{F}_p} \colon \mathbf{F}_p^{c_n^q} \to \mathbf{F}_p^{|E(G_n)|}$. Let $\pi_1 \colon \mathbf{Z}^{c_n^q} \to \mathbf{F}_p^{c_n^q}$, $\pi_2 \colon \mathbf{Z}^{|E(G_n)|} \to \mathbf{F}_p^{|E(G_n)|}$ be the residue class maps. Then $\pi_2 \circ \rho_{\mathbf{Z}} = \rho_{\mathbf{F}_p} \circ \pi_1$. Therefore,

$$\operatorname{rank}_{\mathbf{Z}} \operatorname{Im} \rho_{\mathbf{Z}} \geq \dim_{\mathbf{F}_{\rho}} \operatorname{Im} \rho_{\mathbf{F}_{\rho}}$$
.

Clearly, $\operatorname{rank}_{\mathbf{Z}} \operatorname{Im} \rho_{\mathbf{Z}} = \dim_{\mathbf{Q}} C_{\mathbf{Q}}^{q}(G_{n})$ and $\dim_{\mathbf{F}_{\rho}} \operatorname{Im} \rho_{\mathbf{F}_{\rho}} = \dim_{\mathbf{F}_{\rho}} C_{\mathbf{F}_{\rho}}^{q}(G_{n})$. Thus our lemma follows.

By our lemma, $\beta_{\mathbf{Q}}(\mathcal{G}) \leq \beta_{\mathbf{F}_p}(\mathcal{G})$ hence we finish the proof of our proposition. \square

QUESTION 2.5. Does there exist a graph sequence \mathcal{G} for which $\beta_{\mathbf{Q}}(\mathcal{G}) \neq \beta_{\mathbf{F}_{a}}(\mathcal{G})$ or $\beta_{\mathbf{F}_{a}}(\mathcal{G}) \neq c(\mathcal{G}) - 1$?

Finally we prove Theorem 1.1.

Proof. Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a large girth graph sequence. Then by definition $\beta_K(\mathcal{G}) = e(\mathcal{G}) - 1$. That is, $e(\mathcal{G}) - 1 \leq c(\mathcal{G}) - 1$, hence our theorem follows. \square

Residually finite groups

The goal of this section is to prove Theorem 1.2. Let Γ be a finitely generated residually finite group with a not necessarily symmetric generating system S. Let $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \ldots$, $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$ be a nested sequence of finite index normal subgroups and $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be the graph sequence, where G_n is the (left) Cayley-graph of the finite group Γ/Γ_n with respect

to S. Note that if S' is another generating system and $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ is the associated graph sequence then $\mathcal{H} \simeq \mathcal{G}$.

PROPOSITION 3.1.
$$c(\mathcal{G}) - 1 \le \operatorname{rk} \operatorname{grad} \left\{ \Gamma, \left\{ \Gamma_n \right\}_{n=1}^{\infty} \right\}.$$

Proof. First note that by the Reidemeister-Schreier theorem the groups Γ_n are finitely generated as well [9], moreover if T is a finite generating system of Γ_n , then

$$d_{G_{\tau}^{\Gamma_n}}(x, y) \le L d_{G_{S}^{\Gamma}}(x, y)$$

for any $x, y \in \Gamma_n$, where G_S^{Γ} resp. $G_T^{\Gamma_n}$ are the Cayley-graphs with respect to S resp. to T, and the Lipschitz constant L depends only on S and T.

LEMMA 3.2. For any $k \ge 1$,

$$\frac{d(\Gamma_k)}{|\Gamma:\Gamma_k|} + 1 \ge c(\Gamma).$$

Proof. We use an idea resembling an argument in the proof of Theorem 21.1 of [4]. Let T be a generating system of Γ_k of minimal number of generators. For simplicity we suppose that $T \subset S$. Consider the following graph sequence: $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$, $V(H_n) = \Gamma/\Gamma_n$. If $n \leq k$, let $H_n = G_n$. Set $S_n = \Gamma_k/\Gamma_n$ and let H'_n be the Cayley-graph of S_n with respect to T. Now enumerate the vertices of $V(H_n)\backslash S_n$, $\{x_1,x_2,\ldots,x_{r_n}\}$. For each x_i consider the set of shortest paths in G_n from x_i to the set S_n . Pick the minimal path with respect to the lexicographic ordering. The edges of H_n shall consist of H'_n and the edges of the minimal paths. Define a map $\pi: V(H_n) \to S_n$ in the following way. For each $x_i \in V(H_n)\backslash S_n$ let $\pi(x_i) \in S_n$ be the endpoint of the minimal path from x_i to S_n and let $\pi(x) = x$ if $x \in S_n$. By the lexicographic minimality, the union of the paths form a subforest in G_n having exactly $|V(H_n)\backslash S_n|$ edges.

We claim that $\mathcal{H} \simeq \mathcal{G}$. Since $\mathcal{H} \subset \mathcal{G}$, we only need to prove that $\mathcal{G} \prec \mathcal{H}$. Let n > k, $x, y \in V(G_n)$. Consider the shortest G_n -path from x to y, $\{x_0, x_1, \ldots x_l\}$, $x_0 = x$, $x_l = y$. Let us consider the sequence of vertices $\{\pi(x_0), \pi(x_1), \ldots, \pi(x_l)\}$.

Let $y_1, y_2, \ldots, y_{|\Gamma:\Gamma_k|}$ be a set of coset-representatives with respect to Γ_k . Let t be the maximal word-length of the representatives with respect to S. Then $d_{G_n}(\pi(x), x) \leq t$ for any $x \in V(G_n)$. Therefore, $d_{G_n}(\pi(x_i), \pi(x_{i+1})) \leq 2t + 1$. That is, $d_{H_n}(\pi(x_i), \pi(x_{i+1})) \leq L(2t+1)$, where L is the Lipschitz-constant defined before the statement of our lemma. Consequently,

$$d_{H_n}(x, y) \le L(2t + 1) d_{G_n}(x, y)$$

and therefore $\mathcal{H} \simeq \mathcal{G}$.

For the edge measure of \mathcal{H} we have

$$e(\mathcal{H}) = \liminf_{n \to \infty} \frac{|\Gamma : \Gamma_n| - |\Gamma_k : \Gamma_n| + |E(H'_n)|}{|\Gamma : \Gamma_n|}.$$

The vertex degrees of H'_n are not greater than $2|T|=2d(\Gamma_k)$, also $|S_n|=|\Gamma_k:\Gamma_k|$. Thus

$$c(\mathcal{G}) \leq e(\mathcal{H}) \leq \frac{d(\Gamma_k)}{|\Gamma : \Gamma_k|} + 1$$
.

Hence the lemma follows.

Proposition 3.1 is a straightforward consequence of Lemma 3.2.

Let $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}, S, \mathcal{G}$ be as above. Moreover suppose that Γ is finitely presented. This means that if $\Theta \colon \mathcal{F}_S \to \Gamma$ is the natural map from the free group generated by S to Γ then $\ker \Theta$ is generated by the relations $\{R_1, R_2, \ldots, R_l\}$ as a normal subgroup, that is, if $\Theta(\underline{w}) = 1$ then

$$\underline{w} = \prod_{j=1}^{r_w} \gamma_j R_{ij} \gamma_j^{-1}, \quad \gamma_j \in \mathcal{F}_{\mathcal{S}}.$$

Let $\widetilde{\Sigma}$ be the usual covering CW-complex constructed from $\{R_i\}_{i=1}^l$, the 1-skeleton of $\widetilde{\Sigma}$ is the Cayley-graph of Γ and for each $\gamma \in \Gamma$ and $1 \leq i \leq l$, we add a 2-cell $\sigma_{\gamma,i}$ such that

$$\partial \sigma_{\gamma,i} = \sum_{i=1}^{s_i} (\underline{w}_j \gamma, \underline{w}_{j-1} \gamma),$$

where $R_i = a_{s_i}a_{s_{i-1}}\dots a_2a_1$, $\underline{w}_j = a_ja_{j-1}\dots a_2a_1$, $\underline{w}_0 = 1$. Then $\widetilde{\Sigma}$ is simply connected with a natural Γ -action. Clearly, $\pi_1(\widetilde{\Sigma}/\Gamma_n) = \Gamma_n$. Recall that the group homology space $H_1(\Gamma_n, K)$ is isomorphic to the CW-homology space $H_1(\widetilde{\Sigma}/\Gamma_n, K)$.

LEMMA 3.3. We have

$$\lim_{n\to\infty}\frac{\dim_K H_1(\widetilde{\Sigma}/\Gamma_n,K)}{|\Gamma:\Gamma_n|}=\beta_K(\mathcal{G}).$$

Proof. Consider the homology complex

$$C_2(\widetilde{\Sigma}/\Gamma_n, K) \xrightarrow{\partial_2} C_1(\widetilde{\Sigma}/\Gamma_n, K) \xrightarrow{\partial_1} C_0(\widetilde{\Sigma}/\Gamma_n, K)$$
.

Observe that

$$C_1(\widetilde{\Sigma}/\Gamma_n, K) \simeq \varepsilon_K(G_n)$$
 and $\dim_K C_0(\widetilde{\Sigma}/\Gamma_n, K) = |V(G_n)|$.

Let r be the maximal word-length of a relation R_i . Then $\partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ is generated by cycles of length at most r. On the other hand, for any q > r, the q-cycles are in $\partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ if n is large enough.

Therefore $C_K^q(G_n) = \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K))$ if n is large enough. Consequently,

$$s_K^q(\mathcal{G}) = \liminf_{n \to \infty} \frac{|E(G_n)| - \dim_K \partial_2(C_2(\widetilde{\Sigma}/\Gamma_n, K)) - |V(G_n)|}{|\Gamma : \Gamma_n|}.$$

On the other hand,

$$\begin{split} \frac{\dim_K H_1(\widetilde{\Sigma}/\Gamma_n,K)}{|\Gamma:\Gamma_n|} &= \frac{\dim_K \ker \partial_1 - \dim_K \operatorname{Im} \partial_2}{|\Gamma:\Gamma_n|} \\ &= \frac{|E(G_n)| - \dim_K \partial_2 (C_2(\widetilde{\Sigma}/\Gamma_n,K)) - |V(G_n)| + 1}{|\Gamma:\Gamma_n|} \,. \end{split}$$

Hence the lemma follows.

Now we prove the second part of Theorem 1.2.

PROPOSITION 3.4. Let Γ be a finitely presented residually finite group, $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}, S, \mathcal{G}$ be as above. Then

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \beta_{\mathbf{Q}}^{1}(\Gamma) \le p \operatorname{-grad}\left\{\Gamma, \left\{\Gamma_{n}\right\}_{n=1}^{\infty}\right\} = \beta_{\mathbf{F}_{n}}(\mathcal{G}) \le c(\mathcal{G}) - 1,$$

where $\beta_{(2)}^1(\Gamma)$ is the first L^2 -Betti number of Γ (see [8]).

Proof. By Lemma 3.3, $\beta_{\mathbf{F}_p}(\mathcal{G}) = p\operatorname{-grad}\left\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\right\}$. Also,

$$\beta_{\mathbf{Q}}(\mathcal{G}) = \liminf_{n \to \infty} \frac{\dim_{\mathbf{Q}} H_1(\widetilde{\Sigma}/\Gamma_n, \mathbf{Q})}{|\Gamma : \Gamma_n|}.$$

By the Approximation Theorem of Lück

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{Q}} H_1(\widetilde{\Sigma}/\Gamma_n, \mathbb{Q})}{|\Gamma:\Gamma_n|} = \beta_{(2)}^1(\Gamma).$$

Hence our proposition follows.

QUESTION 3.5. 1. Does there exist a finitely presented residually finite group Γ and a system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ such that

$$\beta_{(2)}^1(\Gamma) \neq p$$
-grad $\left\{\Gamma, \left\{\Gamma_n\right\}_{n=1}^{\infty}\right\}$ or p -grad $\left\{\Gamma, \left\{\Gamma_n\right\}_{n=1}^{\infty}\right\} \neq c(\mathcal{G}) - 1$?

2. Does there exist a finitely generated residually finite group Γ and a system $\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\}$ such that

$$c(\mathcal{G}) - 1 \neq \text{rk grad} \left\{ \Gamma, \left\{ \Gamma_n \right\}_{n=1}^{\infty} \right\} ?$$

4. HYPERFINITE GRAPH SEQUENCES

We say that a graph sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is hyperfinite if for any $\epsilon > 0$ there exists $K_{\epsilon} > 0$, positive integers $\{k_n\}_{n=1}^{\infty}$ and a sequence of partitions of the vertex sets $V(G_n)$

$$A_1^n \cup A_2^n \cup \cdots \cup A_{k_n}^n = V(G_n)$$

such that

- For any $n \ge 1$, $1 \le i \le k_n$, $|A_i^n| \le K_{\epsilon}$.
- If E_n^{ϵ} is the set of edges $(x,y) \in E(G_n)$ such that $x \in A_i$, $y \in A_j$, $x \neq y$, then

$$\liminf_{n\to\infty}\frac{|E_n^{\epsilon}|}{|V(G_n)|}\leq \epsilon.$$

Now we prove Proposition 1.3.

Proof. Suppose that $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is hyperfinite. Let $\mathcal{H}^{\epsilon} = \{H_n^{\epsilon}\}_{n=1}^{\infty}$ be the following graph sequence. The vertex set of H_n^{ϵ} is $V(G_n)$, $E(H_n^{\epsilon})$ is the union of E_n^{ϵ} and a spanning tree for each connected component of the graphs spanned by the vertices of A_i^n , $1 \leq i \leq k_n$. Clearly, $\mathcal{H}^{\epsilon} \simeq \mathcal{G}$ and $|E(H_n^{\epsilon})| \leq |E_n^{\epsilon}| + |V(G_n)|$ thus $e(\mathcal{H}^{\epsilon}) \leq 1 + \epsilon$. Therefore $c(\mathcal{G}) = 1$.

Now we show that for any graph sequence $\mathcal{G}=\{G_n\}_{n=1}^\infty$, $\mathcal{H}=\{H_n\}_{n=1}^\infty$ is hyperfinite where H_n is a spanning tree of G_n . We actually show that a sequence of trees $\mathcal{T}=\{T_n\}_{n=1}^\infty$ is always hyperfinite. Let q be an integer and consider a maximal q-net $L_n^q\subset V(T_n)$. That is, if $x\neq y\in L_n^q$ then $d_{T_n}(x,y)\geq q$ and for any $z\in V(T_n)$ there exists $x\in L_n^q$ such that $d_{T_n}(x,z)\leq q$. Now for each $x\in V(T_n)$ let $\pi(x)$ be one of the vertices $y\in L_n^q$ closest to x. Then $\bigcup_{y\in L_n^q}\pi^{-1}(y)$ is a partition of $V(T_n)$. Clearly $|\pi^{-1}(y)|\geq q$ for any $y\in L_n^q$. Obviously the T_n^y subgraph spanned by the vertices in $\pi^{-1}(y)$ is connected. Thus

$$|E_n^{\epsilon}| \leq |V(T_n)| - (|V(T_n)| - |L_n^q|).$$

Here we used the fact that a connected graph has at least as many edges as the number of its vertices minus one. Obviously, $|L_n^q| \leq \frac{|V(T_n)|}{a}$, therefore

$$\lim_{n\to\infty}\frac{\left|E_n^{\epsilon}\right|}{\left|V(T_n)\right|}\leq \frac{1}{q}.$$

Consequently, the graph sequence T is indeed hyperfinite.

Finally, we prove Theorem 1.4.

Proof. First let Γ be a residually finite non-amenable group with a symmetric generating system S and a nested sequence of finite index normal subgroups $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \ldots$, $\bigcap_{n=1}^\infty \Gamma_n = \{1\}$. Let G_n be the Cayley-graph of Γ/Γ_n with respect to S and G_S^Γ be the Cayley-graph of the group Γ . Since Γ is non-amenable, it has no Følner-exhaustion, consequently there exists a real number $\delta > 0$ such that for each finite subset $F \subset \Gamma$ the number of edges from F to the complement of F is at least $\delta |F|$. Fix an integer m > 0. If n is large enough then for any subset $M \subset \Gamma/\Gamma_n$, $|M| \leq m$ the number of edges from M to its complement must be at least $\delta |M|$. This follows easily form the fact that for any $r \geq 0$, the r-balls in G_n and in G_S^Γ are isometric. This implies that $\mathcal G$ is not hyperfinite.

Now let Γ , $\{\Gamma_n\}_{n=1}^{\infty}$, S, G be as above, but let Γ be amenable. The following lemma is a straightforward consequence of Theorem 2 of [1].

LEMMA 4.1. For any $\omega > 0$, there exist $L_{\omega} > 0$, $M_{\omega} > 0$ and a sequence of family of subsets

$$\{W_n^i\}_{i=1}^{k_n}, \quad W_n^i \subset V(G_n) \quad if \quad n \ge M_\omega$$

such that for any $1 \le i \le k_n$,

- $|W_n^i| \leq L_{\omega}$,
- $|W_n^i \setminus \bigcup_{j \neq i}^{k_n} W_n^j| \ge (1 \omega) |W_n^i|,$
- the number of edges from W_n^i to its complement is at most $\omega |W_n^i|$,
- $|\bigcup_{i=1}^{k_n} W_n^i| \ge (1-\omega)|V(G_n)|$.

Now let $Z_n^i = W_n^i \setminus \bigcup_{i \neq i}^{k_n} W_n^j$ and consider the partition of $V(G_n)$,

$$V(G_n) = \bigcup_{i=1}^{k_n} Z_n^i \cup \bigcup_{i=1}^{l_n} T_n^i,$$

where T_n^i are arbitrary subsets of size at most L_ω . Let E_n^ω be the set of edges $(x,y)\in G_n$ such that their endpoints belong to different subsets in the partition. There are three kinds of edges in E_n^ω :

- Edges with an endpoint in T_n^i . The number of such edges is at most $2|S|(1-(1-\omega)^2)|V(G_n)|$.
- Edges from Z_n^i to the complement of W_n^i , for some $1 \le i \le k_n$. The number of such edges is at most $2|S|\omega(1-\omega)^{-1}|V(G_n)|$.
- Edges from Z_n^i to $W_n^i \setminus Z_n^i$ for some $1 \le i \le k_n$. The number of such edges is at most $2|S|\omega(1-\omega)^{-1}|V(G_n)|$.

Hence

$$\liminf_{n \to \infty} \frac{|E_n^{\omega}|}{|V(G_n)|} \le 2|S| \left((1 - (1 - \omega)^2) + 2\omega (1 - \omega)^{-1} \right).$$

Therefore \mathcal{G} is hyperfinite.

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