Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 53 (2007)

Heft: 1-2

Artikel: Riemannian holonomy and algebraic geometry

Autor: Beauville, Arnaud

Bibliographie

DOI: https://doi.org/10.5169/seals-109541

denote by $H^{p,q} \subset H^{p+q}(X, \mathbf{C})$ the subspace of de Rham cohomology classes of forms of type (p, q); we have $H^{q,p} = \overline{H^{p,q}}$. The fundamental result of Hodge theory is the Hodge decomposition

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q=n} H^{p,q},$$

together with the canonical isomorphisms $H^{p,q} \xrightarrow{\sim} H^q(X, \Omega_X^p)$. In particular,

$$H^2(X, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

with $H^{2,0} \cong H^0(X, \Omega_X^2)$, embedded into $H^2(X, \mathbf{C})$ by associating to a holomorphic form its De Rham class.

To any hermitian metric g on X is associated a real 2-form ω of type $(1,1)$, the *Kähler form,* defined by $\omega(V, W) = g(V, JW)$ for any real vector fields V, W; the metric is Kähler if ω is closed. Then its class in $H^2(X, \mathbf{C})$ is called a Kähler class. The Kähler classes form an open cone in $H^{1,1}_{\mathbf{R}} = H^{1,1} \cap H^2(X, \mathbf{R})$.

Let L be a line bundle on X. The Chern class $c_1(L) \in H^2(X, \mathbf{C})$ is integral, that is comes from $H^2(X, \mathbf{Z})$, and belongs to $H^{1,1}$. Conversely, any integral class in $H^{1,1}$ is the Chern class of some line bundle on X (Lefschetz theorem).

If L is very ample, its Chern class is the pull-back by φ_L of the Chern class of $\mathcal{O}_{\mathbf{P}}(1)$, which is a Kähler class, and therefore $c_1(L)$ is a Kähler class. More generally, if L is ample, some multiple of $c_1(L)$ is a Kähler class, hence also $c_1(L)$. Conversely, the celebrated Kodaira embedding theorem asserts that *a line bundle whose Chern class is Kähler is ample*. As a corollary, we see that *any compact Kähler manifold X with $H^0(X, \Omega_X^2) = 0$ is projective*: we have $H^2(X, \mathbf{C}) = H^{1,1}$, hence the cone of Kähler classes is open in $H^2(X, \mathbf{R})$. Therefore it contains integral classes; by the above results such a class is the first Chern class of an ample line bundle, hence X is projective. More generally, the same argument shows that X is projective whenever the subspace $H^{1,1}$ of $H^2(X, \mathbf{C})$ is defined over $\mathbf{Q}$.

REFERENCES

[A] ATIYAH, M. On analytic surfaces with double points. *Proc. Roy. Soc. London Ser. A 247* (1958), 237–244.

[Ba] BATYREV, V. Birational Calabi-Yau n-folds have equal Betti numbers. *New trends in algebraic geometry (Warwick, 1996),* 1–11. London Math. Soc. Lecture Note Ser. *264,* Cambridge Univ. Press, 1999.

[B1] BEAUVILLE, A. Variétés kählériennes dont la première classe de Chern est nulle. *J. Differential Geom. 18* (1983), 755–782.

[B2] —— Surfaces K3. Exp. 609 du Séminaire Bourbaki. *Astérisque 105–106* (1983), 217–229.

[B3] —— Fano contact manifolds and nilpotent orbits. *Comment. Math. Helv. 73* (1998), 566–583.

[B-D] BEAUVILLE, A. et R. DONAGI. La variété des droites d'une hypersurface cubique de dimension 4. *C.R. Acad. Sc. Paris 301, sér. I* (1985), 703–706.

[Be] BERGER, M. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France 83* (1955), 279–330.

[Bs] BESSE, A. Einstein manifolds. *Ergebnisse der Math. 10.* Springer-Verlag, Berlin, 1999.

[B-G] BROWN, R. and A. GRAY. Riemannian manifolds with holonomy group Spin(9). In: *Differential Geometry, in Honor of K. Yano*, 41–59. Kinokuniya, Tokyo (1972).

[B-Y] BOCHNER, S. and K. YANO. Curvature and Betti numbers. *Annals of Math. Studies 32.* Princeton University Press, 1953.

[Bg1] BOGOMOLOV, F. Hamiltonian Kähler manifolds. *Soviet Math. Dokl. 19* (1978), 1462–1465.

[Bg2] —— On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). *Geom. Funct. Anal. 6* (1996), 612–618.

[Bo] BOURGUIGNON, J.-P. Premières formes de Chern des variétés kählériennes compactes [d'après E. Calabi, T. Aubin et S.T. Yau]. Séminaire Bourbaki, Exp. 507, 1–21. Lecture Notes *710*, Springer, Berlin, 1979.

[C] CALABI, E. On Kähler manifolds with vanishing canonical class. *Algebraic Geometry and Topology (in honor of S. Lefschetz)*, 78–89. Princeton University Press, 1957.

[De] DEBARRE, O. Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles. *C.R. Acad. Sci. Paris 299, sér. I* (1984), 681–684.

[Dm] DEMAILLY, J.-P. On the Frobenius integrability of certain holomorphic p-forms. In: *Complex Geometry (Göttingen, 2000)*, 93–98. Springer, Berlin, 2002.

[F] FOGARTY, J. Algebraic families on an algebraic surface. *Amer. J. Math. 90* (1968), 511–521.

[Fr] FRIEDMAN, R. On threefolds with trivial canonical bundle. *Complex Geometry and Lie theory*, 103–134. Proc. Sympos. Pure Math. *53*, AMS, 1991.

[F-N] FUJIKI, A. and S. NAKANO. Supplement to "On the inverse of monoidal transformation". *Publ. Res. Inst. Math. Sci. 7* (1971-72), 637–644.

[H] HUYBRECHTS, D. Compact hyperkähler manifolds: basic results. *Invent. Math. 135* (1999), 63–113; see also: Erratum, *Invent. Math. 152* (2003), 209–212.

[H-L] HUYBRECHTS, D. and M. LEHN. *The Geometry of Moduli Spaces of Sheaves.* Aspects of Math. *E31.* Vieweg, 1997.

[H-S] HITCHIN, N. and J. SAWON. Curvature and characteristic numbers of hyper-Kähler manifolds. *Duke Math. J. 106* (2001), 599–615.

[J1] JOYCE, D. Compact Riemannian 7-manifolds with holonomy G_2. *J. Differential Geom. 43* (1996), 291–375.

[J2] —— Compact Riemannian 8-manifolds with holonomy Spin(7). *Invent. Math. 123* (1996), 507–552.

[J3] —— Compact manifolds with exceptional holonomy. *Doc. Math. J. DMV* Extra Volume ICM II (1998), 361–370.

[K] KAPRANOV, M. Rozansky-Witten invariants via Atiyah classes. *Compositio Math. 115* (1999), 71–113.

[K-L-S] KALEDIN, D., M. LEHN and C. SORGER. Singular symplectic moduli spaces. *Invent. math. 164* (2006), 591–614.

[KPSW] KEBEKUS, S., T. PETERNELL, A. SOMMESE and J. WIŚNIEWSKI. Projective contact manifolds. *Invent. Math. 142* (2000), 1–15.

[K-N] KOBAYASHI, S. and K. NOMIZU. *Foundations of Differential Geometry*. Wiley-Interscience Publications, 1963.

[Ko] KONTSEVICH, M. Homological algebra of mirror symmetry. *Proc. ICM Zürich 1994*, vol. 1, 120–139. Birkhäuser, Basel, 1995.

[L] LE BRUN, C. Fano manifolds, contact structures, and quaternionic geometry. *Internat. J. Math. 6* (1995), 419-437.

[M] MUKAI, S. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. *Invent. Math. 77* (1984), 101–116.

[N] NAMIKAWA, Y. Counter-example to global Torelli problem for irreducible symplectic manifolds. *Math. Ann. 324* (2002), 841–845.

[Ni] NIEPER, M. Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds. *J. Algebraic Geom. 12* (2003), 715–739.

[O] OLMOS, C. A geometric proof of the Berger holonomy theorem. *Ann. of Math. (2) 161* (2005), 579–588.

[OG1] O'GRADY, K. The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface. *J. Algebraic Geom. 6* (1997), 599–644.

[OG2] —— Desingularized moduli spaces of sheaves on a K3. *J. reine angew. Math. 512* (1999), 49–117.

[OG3] —— A new six-dimensional irreducible symplectic variety. *J. Algebraic Geom. 12* (2003), 435–505.

[P] *Géométrie des surfaces K3 : modules et périodes* (séminaire Palaiseau 81–82). *Astérisque 126* (1985).

[R] DE RHAM, G. Sur la réductibilité d'un espace de Riemann. *Comment. Math. Helv. 26* (1952), 328–344.

[R-W] ROZANSKY, L. and E. WITTEN. Hyper-Kähler geometry and invariants of three-manifolds. *Selecta Math. (N.S.) 3* (1997), 401–458.

[S] SALAMON, S. Quaternionic Kähler manifolds. *Invent. Math. 67* (1982), 143–171.

[Si] SIMONS, J. On the transitivity of holonomy systems. *Ann. of Math. (2) 76* (1962), 213–234.

[S-Y-Z] STROMINGER, A., S. T. YAU and E. ZASLOW. Mirror symmetry is T-duality. *Nuclear Phys. B 479* (1996), 243–259.

[T] TIAN, G. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. Math. aspects of string theory, 629–646. *Adv. Ser. Math. Phys. 1*. World Scientific, Singapore, 1987.

[Va] VAROUCHAS, J. Stabilité de la classe des variétés kählériennes par certains morphismes propres. *Invent. Math. 77* (1984), 117–127.

[V1] VOISIN, C. *Symétrie miroir. Panoramas et Synthèses 2.* SMF, 1996.

[V2] —— On the homotopy types of compact Kähler and complex projective manifolds. *Invent. Math. 157* (2004) 329–343.

[W] WOLF, J. Complex homogeneous contact manifolds and quaternionic symmetric spaces. *J. Math. Mech. 14* (1965), 1033–1047.

[Y] YAU, S. T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math. 31* (1978), 339–411.

(Reçu le 17 janvier 2007)

Arnaud Beauville

Institut Universitaire de France et Laboratoire J.-A. Dieudonné
Université de Nice
Parc Valrose
F-06108 Nice Cedex 2
e-mail: beauville@math.unice.fr