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denote by $H^{p,q} \subset H^{p+q}(X, \mathbf{C})$ the subspace of de Rham cohomology classes of forms of type (p, q); we have $H^{q,p} = \overline{H^{p,q}}$. The fundamental result of Hodge theory is the Hodge decomposition

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q=n} H^{p,q},$$

together with the canonical isomorphisms $H^{p,q} \xrightarrow{\sim} H^q(X, \Omega_X^p)$. In particular,

$$H^2(X, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

with $H^{2,0} \cong H^0(X, \Omega_X^2)$, embedded into $H^2(X, \mathbf{C})$ by associating to a holomorphic form its De Rham class.

To any hermitian metric g on X is associated a real 2-form ω of type $(1,1)$, the *Kähler form,* defined by $\omega(V, W) = g(V, JW)$ for any real vector fields V, W; the metric is Kähler if ω is closed. Then its class in $H^2(X, \mathbf{C})$ is called a Kähler class. The Kähler classes form an open cone in $H^{1,1}_{\mathbf{R}} = H^{1,1} \cap H^2(X, \mathbf{R})$.

Let L be a line bundle on X. The Chern class $c_1(L) \in H^2(X, \mathbf{C})$ is integral, that is comes from $H^2(X, \mathbf{Z})$, and belongs to $H^{1,1}$. Conversely, any integral class in $H^{1,1}$ is the Chern class of some line bundle on X (Lefschetz theorem).

If L is very ample, its Chern class is the pull-back by φ_L of the Chern class of $\mathcal{O}_{\mathbf{P}}(1)$, which is a Kähler class, and therefore $c_1(L)$ is a Kähler class. More generally, if L is ample, some multiple of $c_1(L)$ is a Kähler class, hence also $c_1(L)$. Conversely, the celebrated Kodaira embedding theorem asserts that *a line bundle whose Chern class is Kähler is ample*. As a corollary, we see that *any compact Kähler manifold X with $H^0(X, \Omega_X^2) = 0$ is projective*: we have $H^2(X, \mathbf{C}) = H^{1,1}$, hence the cone of Kähler classes is open in $H^2(X, \mathbf{R})$. Therefore it contains integral classes; by the above results such a class is the first Chern class of an ample line bundle, hence X is projective. More generally, the same argument shows that X is projective whenever the subspace $H^{1,1}$ of $H^2(X, \mathbf{C})$ is defined over $\mathbf{Q}$.

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