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Consider the composite map

$$Z \hookrightarrow X \times \widetilde{A} \to \widetilde{A} \to V, \tag{10.2}$$

where the final step uses that $\widetilde{A}$ is constructed inside of $\mathbf{P}^n_k \times V$. The map (10.2) is dominant, since even $W_P = Z_{x_0} \subseteq Z$ maps birationally onto V, so Z hits the generic point $\eta \in V$ with fiber Z_η that must be integral and have dimension $\dim Z - \dim V = 1$. Thus, the proper map

$$Z \hookrightarrow X \times \widetilde{A} \to X \times V$$

has restriction over X_K that is a proper map $\xi \colon Z_\eta \to X_K$ between integral curves over K. Since X_K is a K-smooth curve, ξ is either constant or finite and flat. The fibers of ξ over the K-points $\{x_0\} \times_{\operatorname{Spec} k} K$ and $\{x_0'\} \times_{\operatorname{Spec} k} K$ of X_K are $(Z_{x_0})_\eta = (W_P)_\eta$ and $(Z_{x_0'})_\eta = (W_{P'})_\eta$, and these are non-empty because $W_P \to V$ and $W_{P'} \to V$ are dominant (even birational) morphisms. Thus, ξ must be finite and flat. Since $W_P \to V$ is birational, so $(W_P)_\eta \to \eta$ is an isomorphism, ξ has degree 1 and thus is an isomorphism. It follows that for some dense open $V^0 \subseteq V$, the restriction of the composite $Z \hookrightarrow X \times \widetilde{A} \to X \times V$ over $X \times V^0$ is an isomorphism.

Hence, we can consider $Z|_{V^0}$ as a section $\mathcal{P}_{V^0} \colon X_{V^0} \to X_{V^0} \times_{V^0} \widetilde{A}_{V^0}$. Restricting this over the generic point η of V^0 and recalling that (by construction of $\widetilde{A}$) the map $\widetilde{A} \to V$ has generic fiber equal to the abelian variety A over η, we arrive at a section $\mathcal{P}_K \colon X_K \to X_K \times A$ over X_K such that $\mathcal{P}_K(\{x_0\}_K) \in A(K)$ is the K-point P that was used to define W_P via closure, and likewise $\mathcal{P}_K(\{x_0'\}_K) \in A(K)$ is P'. It is therefore enough to prove that for *all* $x \in X(k)$, the points $\mathcal{P}_K(x) \in A(K)$ coincide modulo $\operatorname{Tr}_{K/k}(A)(k)$. The argument with Albanese varieties that we used to conclude the proof of the Lang-Néron theorem may now be carried over *verbatim* to prove this final claim. □

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