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Consider the composite map

$$(10.2) \quad Z \hookrightarrow X \times \tilde{A} \rightarrow \tilde{A} \rightarrow V,$$

where the final step uses that \tilde{A} is constructed inside of $\mathbf{P}_k^n \times V$. The map (10.2) is dominant, since even $W_P = Z_{x_0} \subseteq Z$ maps birationally onto V , so Z hits the generic point $\eta \in V$ with fiber Z_η that must be integral and have dimension $\dim Z - \dim V = 1$. Thus, the proper map

$$Z \hookrightarrow X \times \tilde{A} \rightarrow X \times V$$

has restriction over X_K that is a proper map $\xi: Z_\eta \rightarrow X_K$ between integral curves over K . Since X_K is a K -smooth curve, ξ is either constant or finite and flat. The fibers of ξ over the K -points $\{x_0\} \times_{\text{Spec } k} K$ and $\{x'_0\} \times_{\text{Spec } k} K$ of X_K are $(Z_{x_0})_\eta = (W_P)_\eta$ and $(Z_{x'_0})_\eta = (W_{P'})_\eta$, and these are non-empty because $W_P \rightarrow V$ and $W_{P'} \rightarrow V$ are dominant (even birational) morphisms. Thus, ξ must be finite and flat. Since $W_P \rightarrow V$ is birational, so $(W_P)_\eta \rightarrow \eta$ is an isomorphism, ξ has degree 1 and thus is an isomorphism. It follows that for some dense open $V^0 \subseteq V$, the restriction of the composite $Z \hookrightarrow X \times \tilde{A} \rightarrow X \times V$ over $X \times V^0$ is an isomorphism.

Hence, we can consider $Z|_{V^0}$ as a section $\mathcal{P}_{V^0}: X_{V^0} \rightarrow X_{V^0} \times_{V^0} \tilde{A}_{V^0}$. Restricting this over the generic point η of V^0 and recalling that (by construction of \tilde{A}) the map $\tilde{A} \rightarrow V$ has generic fiber equal to the abelian variety A over η , we arrive at a section $\mathcal{P}_K: X_K \rightarrow X_K \times A$ over X_K such that $\mathcal{P}_K(\{x_0\}_K) \in A(K)$ is the K -point P that was used to define W_P via closure, and likewise $\mathcal{P}_K(\{x'_0\}_K) \in A(K)$ is P' . It is therefore enough to prove that for *all* $x \in X(k)$, the points $\mathcal{P}_K(x) \in A(K)$ coincide modulo $\text{Tr}_{K/k}(A)(k)$. The argument with Albanese varieties that we used to conclude the proof of the Lang-Néron theorem may now be carried over *verbatim* to prove this final claim. \square

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