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Thus the second smallest lattice is given by the maximal order with $D = -4$ (the square lattice) and the third and fourth smallest lattices by $D = -7$ and $D = -15$ respectively.

REMARK 3. The inequality (5.5) is quite subtle. Let $N_k = 2 \cdot 3 \cdots p_k$ be the product of the first k primes, then if the Riemann Hypothesis is true (5.5) is false for every integer n with $n = N_k$. On the other hand, if the Riemann Hypothesis is false then there are infinitely many integers k for which $n = N_k$ does satisfy (5.5). See Nicolas [23] for a proof of this interesting result.

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REFERENCES

[1] BERNAYS, P. Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante. Dissertation, Göttingen, 1912. http://www.math.uni-bielefeld.de/~rehmann/DML/

[2] BERNDT, B. *Ramanujan's Notebooks*. Part IV. Springer-Verlag, New York, 1994.

[3] BERNDT, B. and K. ONO. Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary. The Andrews Festschrift (Maratea, 1998). *Sém. Lothar. Combin. 42* (1999), Art. B42c.

[4] BLICHFELDT, B. The minimum values of positive quadratic forms in six, seven and eight variables. *Math. Z. 39* (1935), 1–15.

[5] BLOMER, V. Binary quadratic forms with large discriminants and sums of two squareful numbers. *J. Reine Angew. Math. 569* (2004), 213–234.

[6] —— Binary quadratic forms with large discriminants and sums of two squareful integers II. *J. London Math. Soc. (2) 71* (2005), 69–84.

[7] BLOMER, V. and A. GRANVILLE. Estimates for representation numbers of quadratic forms. *Duke Math. J. 135* (2006), 261–302.

[8] CHOIE, Y.-J., N. LICHIARDOPOL, P. MOREE and P. SOLÉ. On Robin's criterion for the Riemann Hypothesis. http://front.math.ucdavis.edu/math.NT/0604314 and to appear in *J. Théor. Nombres Bordeaux*.

[9] CONWAY, J. and N. SLOANE. Lattices with few distances. *J. Number Theory 39* (1991), 75–90.

[10] COX, D. *Primes of the Form* $x^2 + ny^2$. Wiley, New York, 1989.

[11] FINCH, S. *Mathematical Constants*. Encyclopedia of Mathematics and its Applications *94*. Cambridge University Press, Cambridge, 2003.

[12] FOMENKO, O. Distribution of values of Fourier coefficients of modular forms of weight 1. *J. Math. Sci. (New York) 89* (1998), 1050–1071.

[13] IWANIEC, H. The half dimensional sieve. *Acta Arith. 29* (1976), 69–95.

[14] JAMES, R. The distribution of integers represented by quadratic forms. *Amer. J. Math. 60* (1938), 737–744.

[15] KAPLAN, P. and WILLIAMS, K. The genera representing a positive integer. *Acta Arith. 102* (2002), 353–361.

[16] KÜHNLEIN, S. Partial solution of a conjecture of Schmutz. *Arch. Math. (Basel) 67* (1996), 164–172.

[17] LANDAU, E. Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate. *Arch. der Math. und Phys. (3) 13* (1908), 305–312.

[18] MOREE, P. Approximation of singular series and automata. *Manuscripta Math. 101* (2000), 385–399.

[19] —— Chebyshev's bias for composite numbers with restricted prime divisors. *Math. Comp. 73* (2004), 425–449.

[20] —— On some claims in Ramanujan's 'unpublished' manuscript on the partition and tau functions. *Ramanujan J. 8* (2004), 317–330.

[21] MOREE, P. and J. CAZARAN. On a claim of Ramanujan in his first letter to Hardy. *Exposition. Math. 17* (1999), 289–311.

[22] MOREE, P. and H. TE RIELE. The hexagonal versus the square lattice. *Math. Comp. 73* (2004), 451–473.

[23] NICOLAS, J. Petites valeurs de la fonction d'Euler. *J. Number Theory 17* (1983), 375–388.

[24] ODONI, R. Representations of algebraic integers by binary quadratic forms and norm forms from full modules of extension fields. *J. Number Theory 10* (1978), 324–333.

[25] —— The distribution of integral and prime-integral values of systems of full-norm polynomials and affine-decomposable polynomials. *Mathematika 26* (1979), 80–87.

[26] —— A problem of Erdős on sums of two squarefull numbers. *Acta Arith. 39* (1981), 145–162.

[27] PALL, G. The distribution of integers represented by binary quadratic forms. *Bull. Amer. Math. Soc. 49* (1943), 447–449.

[28] REID, C. *Hilbert*. Springer-Verlag, New York-Berlin, 1970.

[29] RIEGER, G. Zum Satz von Landau über die Summe aus zwei Quadraten. *J. Reine Angew. Math. 244* (1970), 198–200.

[30] SCHARLAU, W. and H. OPOLKA. *Von Fermat bis Minkowski. Eine Vorlesung über Zahlentheorie und ihre Entwicklung*. Springer-Verlag, Berlin-New York, 1980.

[31] SCHMUTZ, P. Arithmetic groups and the length spectrum of Riemann surfaces. *Duke Math. J. 84* (1996), 199–215.

[32] SCHMUTZ SCHALLER, P. Geometry of Riemann surfaces based on closed geodesics. *Bull. Amer. Math. Soc. (N.S.) 35* (1998), 193–214.

[33] SERRE, J.-P. Divisibilité de certaines fonctions arithmétiques. *L'Enseignement Math. (2) 22* (1976), 227–260.

[34] SHANKS, D. The second-order term in the asymptotic expansion of $B(x)$. *Math. Comp. 18* (1964), 75–86.

[35] SHANKS, D. and L. SCHMID. Variations on a theorem of Landau. I. *Math. Comp. 20* (1966), 551–569.

[36] SMITH, W. Few-distance sets and the second Erdős number. *Unpublished preprint*, 1990.

[37] SUN, Z. and K. WILLIAMS. On the number of representations of n by $ax^2 + bxy + cy^2$. *Acta Arith. 122* (2006), 101–171.

[38] WILLIAMS, K. Note on integers representable by binary quadratic forms. *Canad. Math. Bull. 18* (1975), 123–125.

[39] WIRSING, E. Das asymptotische Verhalten von Summen über multiplikative Funktionen. *Math. Ann. 143* (1961), 75–102.

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