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**Autor:** Ovsienko, Valentin  
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## VECTOR FIELDS IN THE PRESENCE OF A CONTACT STRUCTURE

by Valentin OVSIENKO

**ABSTRACT.** We consider the Lie algebra of all vector fields on a contact manifold as a module over the Lie subalgebra of contact vector fields. This module is split into a direct sum of two submodules: the contact algebra itself and the space of tangent vector fields. We study the geometric nature of these two modules.

### 1. INTRODUCTION

Let  $M$  be a smooth manifold and  $\text{Vect}(M)$  the Lie algebra of all smooth vector fields on  $M$ . We consider the case when  $M$  is  $(2n + 1)$ -dimensional and can be equipped with a contact structure. For instance, if  $\dim M = 3$ , and  $M$  is compact and orientable, then a famous theorem of 3-dimensional topology states that there is always a contact structure on  $M$ .

Let  $\text{CVect}(M)$  be the Lie algebra of smooth vector fields on  $M$  preserving the contact structure. This Lie algebra naturally acts on  $\text{Vect}(M)$  (by Lie bracket). We will study the structure of  $\text{Vect}(M)$  as a  $\text{CVect}(M)$ -module. First, we observe that  $\text{Vect}(M)$  is split, as a  $\text{CVect}(M)$ -module, into a direct sum of two submodules:

$$\text{Vect}(M) \cong \text{CVect}(M) \oplus \text{TVect}(M),$$

where  $\text{TVect}(M)$  is the space of vector fields tangent to the contact distribution. Note that the latter space is a  $\text{CVect}(M)$ -module but not a Lie subalgebra of  $\text{Vect}(M)$ .

The main purpose of this paper is to study the above two spaces geometrically. The most important notion for us is that of *invariance*. All the maps and isomorphisms we consider are invariant with respect to the group of contact diffeomorphisms of  $M$ . Since we consider only local maps, this is equivalent to invariance with respect to the action of the Lie algebra  $\text{CVect}(M)$ .

It is known, see [5, 6], that the adjoint action of  $\text{CVect}(M)$  has the following geometric interpretation:

$$\text{CVect}(M) \cong \mathcal{F}_{-\frac{1}{n+1}}(M),$$

where  $\mathcal{F}_{-\frac{1}{n+1}}(M)$  is the space of tensor densities of degree  $-\frac{1}{n+1}$  on  $M$ , that is, of sections of the line bundle

$$\left| \bigwedge^{2n+1} T^*M \right|^{-\frac{1}{n+1}} \rightarrow M.$$

In particular, this provides the existence of a nonlinear invariant functional on  $\text{CVect}(M)$  defined on the contact vector fields with nowhere vanishing contact Hamiltonians.

The analogous interpretation of  $\text{TVect}(M)$  is more complicated:

$$\text{TVect}(M) \cong \Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M),$$

where  $\Omega_0^2(M)$  is the space of 2-forms on  $M$  vanishing on the contact distribution. Here and below the tensor products are defined over  $C^\infty(M)$ .

We study the relations between  $\text{TVect}(M)$  and  $\text{CVect}(M)$ . We prove the existence of a non-degenerate skew-symmetric invariant bilinear map

$$\mathcal{B}: \text{TVect}(M) \wedge \text{TVect}(M) \rightarrow \text{CVect}(M)$$

that measures the non-integrability, i.e., the failure of the Lie bracket of two tangent vector fields to remain tangent.

In order to provide explicit formulæ, we introduce a notion of Heisenberg structure on  $M$ . Usually, to write explicit formulæ in contact geometry, one uses the Darboux coordinates. However, this is not the best way to proceed (as already noticed in [4]). The Heisenberg structure provides a universal expression for a contact vector field and its actions.

## 2. CONTACT AND TANGENT VECTOR FIELDS

In this section we recall the basic definitions of contact geometry. We then prove our first statement on a decomposition of the Lie algebra of all smooth vector fields viewed as a module over the Lie algebra of contact vector fields.

### 2.1 MAIN DEFINITIONS

Let  $M$  be a  $(2n + 1)$ -dimensional manifold. A contact structure on  $M$  is a codimension 1 distribution  $\xi$  which is completely non-integrable. The distribution  $\xi$  can be defined (locally) as the kernel of a differential 1-form  $\alpha$  defined up to multiplication by a nowhere vanishing function. Complete non-integrability means that

$$(1) \quad \Omega := \alpha \wedge (d\alpha)^n \neq 0$$

everywhere on  $M$ . The above condition is also equivalent to the fact that the restriction  $d\alpha|_{\xi}$  to any contact hyperplane is a non-degenerate 2-form. In particular,  $\ker d\alpha$  is one-dimensional. Note that if  $M$  is orientable and the contact structure is coorientable, then the form  $\alpha$  can be globally defined on  $M$ ; the form  $\Omega$  is then a volume form.

A vector field  $X$  on  $M$  is a *contact vector field* if it preserves the contact distribution  $\xi$ . In terms of contact forms this means that for every contact form  $\alpha$ , the Lie derivative of  $\alpha$  with respect to  $X$  is proportional to  $\alpha$ :

$$(2) \quad L_X \alpha = f_X \alpha,$$

where  $f_X \in C^\infty(M)$ . The space of all contact vector fields is a Lie algebra that we denote by  $\text{CVect}(M)$ .

Let us now fix a contact form  $\alpha$ . A contact vector field  $X$  is called *strictly contact* if it preserves  $\alpha$ , in other words, if  $f_X = 0$  everywhere on  $M$ . Strictly contact vector fields form a Lie subalgebra of  $\text{CVect}(M)$ . There is one particular strictly contact vector field  $Z$  called the *Reeb field* (or characteristic vector field). It is defined by the following two properties:

$$Z \in \ker d\alpha, \quad \alpha(Z) \equiv 1.$$

We will also consider the space,  $\text{TVect}(M)$ , of vector fields *tangent* to the contact distribution. That this space is not a Lie subalgebra of  $\text{Vect}(M)$  follows from non-integrability of the contact distribution.

## 2.2 THE DECOMPOSITION OF $\text{Vect}(M)$

Let  $\text{Vect}(M)$  be the Lie algebra of all smooth vector fields on  $M$ . The Lie bracket defines a natural action of  $\text{CVect}(M)$  on  $\text{Vect}(M)$ . In particular, the Lie bracket of a contact vector field with a tangent vector field is again a tangent vector field. Therefore,  $\text{TVect}(M)$  is a module over  $\text{CVect}(M)$ .

PROPOSITION 2.1. *The space  $\text{Vect}(M)$  is split into a direct sum of two  $\text{CVect}(M)$ -modules :*

$$\text{Vect}(M) \cong \text{CVect}(M) \oplus \text{TVect}(M).$$

*Proof.* Both spaces on the right hand side are  $\text{CVect}(M)$ -modules. It then remains to check that every vector field can be uniquely decomposed into a sum of a contact vector field and a tangent vector field.

Given a vector field  $X$ , there exists a tangent vector field  $Y$  such that  $X - Y$  is contact. Indeed, consider a 1-form  $\beta = L_X\alpha$  and its restriction  $\beta|_\xi$  to a contact hyperplane  $\xi$ . If  $Y$  is a tangent vector field then  $L_Y\alpha = i_Y(d\alpha)$ . Since  $d\alpha$  is non-degenerate on  $\xi$ , there exists for any 1-form  $\beta$  a tangent field  $Y$  such that  $i_Y(d\alpha)|_\xi = \beta|_\xi$ . This means that  $X - Y$  is contact.

Furthermore, the intersection of  $\text{CVect}(M)$  and  $\text{TVect}(M)$  is zero. Indeed, let  $X$  be a non-zero vector field which is contact and tangent at the same time. Then  $L_X\alpha = f\alpha$  for some function  $f$  and  $L_X\alpha = i_X(d\alpha)$ . Since  $\ker f\alpha$  contains  $\xi = \ker \alpha$  while the restriction  $d\alpha|_\xi$  is non-degenerate, this is a contradiction.  $\square$

## 3. THE ADJOINT REPRESENTATION OF $\text{CVect}(M)$

In this section we study the action of  $\text{CVect}(M)$  on itself.

### 3.1 FIXING A CONTACT FORM: CONTACT HAMILTONIANS

Let  $M$  be orientable; fix a contact form  $\alpha$  on  $M$ . Every contact vector field  $X$  is then characterized by a function

$$H = \alpha(X).$$

This is a one-to-one correspondence between  $\text{CVect}(M)$  and the space  $C^\infty(M)$  of smooth functions on  $M$ , see e.g. [1]. We can denote the contact vector field corresponding to  $H$  by  $X_H$ . The function  $H$  is called the contact Hamiltonian of  $X_H$ .

EXAMPLE 3.1. The contact Hamiltonian of the Reeb field  $Z$  is the constant function  $H \equiv 1$ . Note also that the function  $f_X$  in (2) is given by the derivative  $f_{X_H} = Z(H)$ .

The Lie algebra  $\text{CVect}(M)$  is then identified with  $C^\infty(M)$  equipped with the *Lagrange bracket* defined by  $X_{\{H_1, H_2\}} := [X_{H_1}, X_{H_2}]$ . One checks that

$$(3) \quad \{H_1, H_2\} = X_{H_1}(H_2) - Z(H_1)H_2.$$

The formula expresses the adjoint representation of  $\text{CVect}(M)$  in terms of contact Hamiltonians. The second term on the right hand side shows that this action is different from the natural action of  $\text{CVect}(M)$  on  $C^\infty(M)$ . Let us now clarify the geometric meaning of this action.

### 3.2 SPACE OF TENSOR DENSITIES

Let  $V$  be a vector space of dimension  $d$  and  $\lambda$  an arbitrary real number. A  $\lambda$ -density on  $V$  is a function  $\phi: \bigwedge^d V \setminus \{0\} \rightarrow \mathbf{R}$  homogeneous of degree  $\lambda$ , that is, such that

$$\phi(\kappa w) = |\kappa|^\lambda \phi(w)$$

for all  $\kappa \in \mathbf{R} \setminus \{0\}$  and  $w \in \bigwedge^d V \setminus \{0\}$ . This is a one-dimensional vector space that we denote by  $\mathbf{F}_\lambda(V)$ .

Let  $M$  be a smooth manifold of dimension  $d$ ; consider the determinant bundle  $\bigwedge^d TM \rightarrow M$ .

DEFINITION 3.2. A *tensor density* of degree  $\lambda \in \mathbf{R}$  on  $M$  is a smooth function on the complement of the zero section  $\bigwedge^d TM \setminus M$ , homogeneous of degree  $\lambda$ . The space of tensor densities of degree  $\lambda$  on  $M$  will be denoted by  $\mathcal{F}_\lambda(M)$ .

In other words, a tensor density of degree  $\lambda$  on  $M$  is a section of the line bundle  $\mathbf{F}_\lambda(TM)$ , i.e. a field of  $\lambda$ -densities on the tangent space. Equivalently, consider the line bundle  $\bigwedge^d T^*M \rightarrow M$ . Then, the line bundle  $|\bigwedge^d T^*M|^\lambda$  is well defined for every  $\lambda \in \mathbf{R}$  and naturally isomorphic to  $\mathbf{F}_\lambda(TM)$ . It worth noticing that the bundle  $\mathbf{F}_\lambda(TM)$  is *trivial* for any  $M$  and all  $\lambda$ .

Every space  $\mathcal{F}_\lambda(M)$  is naturally a module over the Lie algebra  $\text{Vect}(M)$ . Let us give here without proof the basic properties of these modules:

- The space  $\mathcal{F}_0(M)$  is simply  $C^\infty(M)$ .
- The space  $\mathcal{F}_1(M)$  is isomorphic as a  $\text{Vect}(M)$ -module to the space  $\Omega_d(M)$  of differential  $d$ -forms if and only if  $M$  is orientable. If  $\Omega$  is a volume form on  $M$ , then one represents tensor densities in the form

$$\varphi = f \Omega^\lambda,$$

where  $f$  is a function.

- The  $\text{Vect}(M)$ -modules  $\mathcal{F}_\lambda(M)$  and  $\mathcal{F}_\mu(M)$  are isomorphic if and only if  $\lambda = \mu$ .
- If  $M$  is compact then there is an invariant functional

$$(4) \quad \int_M : \mathcal{F}_1(M) \rightarrow \mathbf{R}.$$

More generally, there is an invariant pairing

$$\langle \mathcal{F}_\lambda(M), \mathcal{F}_{1-\lambda}(M) \rangle \rightarrow \mathbf{R}$$

given by the integration of the product of tensor densities.

To summarize, the 1-parameter family of spaces  $\mathcal{F}_\lambda(M)$  can be viewed as a deformation of the natural  $\text{Vect}(M)$ -action on the space of functions. We refer to [3] for more information on modules of tensor densities and invariant differential operators on these modules.

### 3.3 TENSOR DENSITIES ON A CONTACT MANIFOLD

Let now  $M$  be a contact manifold of dimension  $d = 2n + 1$ . In this case, there is one more way to define tensor densities. Consider the  $(2n + 2)$ -dimensional submanifold  $S$  of the cotangent bundle  $T^*M \setminus M$  that consists of all non-zero covectors vanishing on the contact distribution  $\xi$ . The restriction to  $S$  of the canonical symplectic structure on  $T^*M$  defines a symplectic structure on  $S$ . The manifold  $S$  is called the *symplectization* of  $M$  (cf. [1, 2]). Clearly  $S$  is a line bundle over  $M$ , its sections are the 1-forms on  $M$  vanishing on  $\xi$ . Note that, in the case where  $M$  is orientable,  $S$  is a trivial line bundle over  $M$ .

There is a natural lift of  $\text{CVect}(M)$  to  $S$ . Indeed, a vector field  $X$  on  $M$  can be lifted to  $T^*M$ , and, if  $X$  is contact, then it preserves the subbundle  $S$ . The space of sections  $\text{Sec}(S)$  is therefore a  $\text{CVect}(M)$ -module.

The sections of the bundle  $S$  can be viewed as tensor densities of degree  $\frac{1}{n+1}$  on  $M$ .

PROPOSITION 3.3. *There is a natural isomorphism of  $\text{CVect}(M)$ -modules*

$$\text{Sec}(S) \cong \mathcal{F}_{\frac{1}{n+1}}(M).$$

*Proof.* A section of  $S$  is a 1-form on  $M$  vanishing on the contact distribution. For every contact vector field  $X$  and a volume form  $\Omega$  as in (1) one has

$$L_X \Omega = (n+1)f_X \Omega.$$

The Lie derivative of a tensor density of degree  $\lambda$  is then given by

$$L_X(f \Omega^\lambda) = (X(f) + \lambda(n+1)f_X f) \Omega^\lambda.$$

The result follows from formula (2).  $\square$

One can now represent tensor densities in terms of a contact form:  $\varphi = f \alpha^{(n+1)\lambda}$ .

### 3.4 CONTACT HAMILTONIAN AS A TENSOR DENSITY

In this section we identify the algebra  $\text{CVect}(M)$  with a space of tensor densities of degree  $-\frac{1}{n+1}$  on  $M$ ; the adjoint action is simply a Lie derivative on this space. The result of this section is known (see [5] and [6], Section 7.5) and given here for the sake of completeness.

Let us define a different version of contact Hamiltonian of a contact vector field  $X$  as a tensor density on  $M$  of degree  $-\frac{1}{n+1}$ :

$$\mathcal{H} := \alpha(X) \alpha^{-1}.$$

An important feature of this definition is that it is independent of the choice of  $\alpha$ . Let us denote the corresponding contact vector field by  $X_{\mathcal{H}}$ .

The space  $\mathcal{F}_{-\frac{1}{n+1}}(M)$  is now identified with  $\text{CVect}(M)$ . Moreover, the Lie bracket of contact vector fields corresponds to the Lie derivative.

PROPOSITION 3.4. *The adjoint representation of  $\text{CVect}(M)$  is isomorphic to  $\mathcal{F}_{-\frac{1}{n+1}}(M)$ .*

*Proof.* The Lagrange bracket coincides with a Lie derivative:

$$(5) \quad \{\mathcal{H}_1, \mathcal{H}_2\} = L_{X_{\mathcal{H}_1}}(\mathcal{H}_2).$$

This formula is equivalent to (3).  $\square$

Geometrically speaking, a contact Hamiltonian is not a function but rather a tensor density of degree  $-\frac{1}{n+1}$ .



### 3.5 INVARIANT FUNCTIONAL ON $\text{CVect}(M)$

Assume  $M$  is compact and orientable; fix a contact form  $\alpha$  and the corresponding volume form  $\Omega = \alpha \wedge d\alpha^n$ . The geometric interpretation of the adjoint action of  $\text{CVect}(M)$  implies the existence of an invariant (non-linear) functional on  $\text{CVect}(M)$ .

Let  $\text{CVect}^*(M)$  be the set of contact vector fields with nowhere vanishing contact Hamiltonians (i.e., the corresponding contact Hamiltonian has no zeroes on  $M$ ). This is an invariant open subset of  $\text{CVect}(M)$ .

**COROLLARY 3.5.** *The functional on  $\text{CVect}^*(M)$  defined by*

$$\mathcal{I}: X_H \mapsto \int_M H^{-(n+1)} \Omega$$

*is invariant. This functional is independent of the choice of the contact form.*

*Proof.* Consider is a contact vector field  $X_F$ , then according to (3) one has

$$L_{X_F}(H^{-(n+1)}) = X_F(H^{-(n+1)}) + (n+1)Z(F)H^{-(n+1)},$$

so that the quantity  $H^{-(n+1)} \Omega$  is a well defined element of the space  $\mathcal{F}_1(M)$ . The functional  $\mathcal{I}$  is then given by the invariant functional (4).

Furthermore, choose a different contact form  $\alpha' = f\alpha$  and the corresponding volume form  $\Omega' = f^{n+1}\Omega$ . The contact Hamiltonian of the vector field  $X_H$  with respect to the contact form  $\alpha'$  is the function  $H' = \alpha'(X_H) = fH$ . Hence,  $H'^{-(n+1)}\Omega' = H^{-(n+1)}\Omega$  so that the functional  $\mathcal{I}$  is, indeed, independent of the choice of the contact form.  $\square$

## 4. THE STRUCTURE OF $\text{TVect}(M)$

In this section we study the structure of the space of tangent vector fields  $\text{TVect}(M)$  viewed as a  $\text{CVect}(M)$ -module.

### 4.1 A GEOMETRIC REALIZATION

Let us start with a geometric realization of the  $\text{CVect}(M)$ -module structure on  $\text{TVect}(M)$  which is quite similar to that of Section 3.4.

Let  $\Omega_0^2(M)$  be the space of 2-forms on  $M$  vanishing on the contact distribution. In other words, elements of  $\Omega_0^2(M)$  are proportional to  $\alpha$ :

$$\omega = \alpha \wedge \beta,$$

where  $\beta$  is an arbitrary 1-form.

The following statement is similar to Proposition 3.4.

THEOREM 4.1. *There is an isomorphism of  $\text{CVect}(M)$ -modules*

$$\text{TVect}(M) \cong \Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M),$$

where the tensor product is defined over  $C^\infty(M)$ .

*Proof.* Let  $M$  be orientable; fix a contact form  $\alpha$  on  $M$ . Consider a linear map from  $\text{TVect}(M)$  to the space  $\Omega_0^2(M)$  that associates to a tangent vector field  $X$  the 2-form

$$\langle X, \alpha \wedge d\alpha \rangle = -\alpha \wedge i_X d\alpha.$$

This map is bijective since the restriction  $d\alpha|_\xi$  of the 2-form  $d\alpha$  to the contact hyperplane  $\xi$  is non-degenerate.

However, the above map depends on the choice of the contact form and, therefore, cannot be  $\text{CVect}(M)$ -invariant. In order to make this map independent of the choice of  $\alpha$ , one defines the map

$$(6) \quad X \mapsto \langle X, \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}$$

with values in  $\Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M)$ . Note that the term  $\alpha^{-2}$  on the right hand side is a well defined element of the space of tensor densities  $\mathcal{F}_{-\frac{2}{n+1}}(M)$ , see Section 3.3.

It remains to check the  $\text{CVect}(M)$ -invariance of the map (6). Let  $X_H$  be a contact vector field; one has

$$\begin{aligned} L_{X_H} (\langle X, \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}) &= \langle [X_H, X], \alpha \wedge d\alpha \rangle \otimes \alpha^{-2} \\ &\quad + \langle X, f_X \alpha \wedge d\alpha + \alpha \wedge df_X \alpha \rangle \otimes \alpha^{-2} \\ &\quad - \langle X, \alpha \wedge d\alpha \rangle \otimes (2f_X \alpha^{-2}) \\ &= \langle [X_H, X], \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}. \end{aligned}$$

Hence the result.  $\square$

The isomorphism (6) identifies the  $\text{CVect}(M)$ -action on  $\text{TVect}(M)$  by Lie bracket with the usual Lie derivative. It is natural to say that this map defines an analog of contact Hamiltonian of a tangent vector field.

#### 4.2 A SKEW-SYMMETRIC PAIRING ON $\text{TVect}(M)$ OVER $\text{CVect}(M)$

There exists an invariant skew-symmetric bilinear map from  $\text{TVect}(M)$  to  $\text{CVect}(M)$  that can be understood as a “symplectic structure” on the space  $\text{TVect}(M)$  over  $\text{CVect}(M)$ .

**THEOREM 4.2.** *There exists a non-degenerate skew-symmetric invariant bilinear map*

$$\mathcal{B}: \text{TVect}(M) \wedge \text{TVect}(M) \rightarrow \text{CVect}(M),$$

where the  $\wedge$ -product is defined over  $C^\infty(M)$ .

*Proof.* Assume first that  $M$  is orientable and fix the contact form  $\alpha$ . Given 2 tangent vector fields  $X$  and  $Y$ , consider the function

$$H_{X,Y} = \langle X \wedge Y, d\alpha \rangle.$$

Define first a bilinear map  $B$  from  $\text{TVect}(M)$  to  $C^\infty(M)$  by

$$(7) \quad B_\alpha: X \wedge Y \mapsto H_{X,Y}.$$

The definition of the function  $H_{X,Y}$  and thus of the map  $B_\alpha$  depends on the choice of  $\alpha$ . Our task is to understand it as a map with values in  $\text{CVect}(M)$  which is independent of the choice of the contact form. This will, in particular, extend the definition to the case where  $M$  is not orientable.

It turns out that the above function  $H_{X,Y}$  is a well defined contact Hamiltonian.

**LEMMA 4.3.** *Choose a different contact form  $\alpha' = f\alpha$ , then*

$$H'_{X,Y} = f H_{X,Y}.$$

*Proof.* By definition,

$$H'_{X,Y} = \langle X \wedge Y, d\alpha' \rangle = f \langle X \wedge Y, d\alpha \rangle + \langle X \wedge Y, df \wedge \alpha \rangle = f H_{X,Y}$$

since the second term vanishes.  $\square$

We observe that the function  $H_{X,Y}$  depends on the choice of  $\alpha$  precisely in the same way as a contact Hamiltonian (cf. Section 3.1). It follows that the bilinear map

$$(8) \quad \mathcal{B}: X \wedge Y \mapsto H_{X,Y} \alpha^{-1}$$

with values in  $\mathcal{F}_{-\frac{1}{n+1}} \cong \text{CVect}(M)$  (cf. Section 3.4) is well defined and independent of the choice of  $\alpha$ .

It remains to check that the constructed map (8) is  $\text{CVect}(M)$ -invariant. This can be done directly but also follows from

PROPOSITION 4.4. *The Lie bracket of two tangent vector fields  $X, Y \in \text{TVect}(M)$  is of the form*

$$(9) \quad [X, Y] = \mathcal{B}(X, Y) + (\text{tangent vector field}).$$

*Proof.* Consider the decomposition from Proposition 2.1 applied to the Lie bracket  $[X, Y]$ . The “non-tangent” component of  $[X, Y]$  is a contact vector field with contact Hamiltonian  $\alpha([X, Y])$ . One has

$$i_{[X, Y]}\alpha = (L_X i_Y - i_Y L_X)\alpha = -i_Y L_X \alpha = -i_Y i_X d\alpha = H_{X, Y}.$$

The result follows.  $\square$

Theorem 4.2 is proved.  $\square$

Proposition 4.4 is an alternative definition of  $\mathcal{B}$ : the map  $\mathcal{B}$  measures the failure of the Lie bracket of two tangent vector fields to remain tangent.

## 5. HEISENBERG STRUCTURES

In order to investigate the structure of  $\text{TVect}(M)$  as a  $\text{CVect}(M)$ -module in more detail, we will write explicit formulæ for the  $\text{CVect}(M)$ -action.

We assume that there is an action of the Heisenberg Lie algebra  $\mathfrak{h}_n$  on  $M$ , such that the center acts by the Reeb field while the generators are tangent to the contact structure. We then say that  $M$  is equipped with the Heisenberg structure. Existence of a globally defined Heisenberg structure is a strong condition on  $M$ , however, locally such structure always exists.

### 5.1 DEFINITION OF A HEISENBERG STRUCTURE

Recall that the Heisenberg Lie algebra  $\mathfrak{h}_n$  is a nilpotent Lie algebra of dimension  $2n+1$  with the basis  $\{a_1, \dots, a_n, b_1, \dots, b_n, z\}$  and the commutation relations

$$[a_i, b_j] = \delta_{ij} z, \quad [a_i, a_j] = [b_i, b_j] = [a_i, z] = [b_i, z] = 0, \quad i, j = 1, \dots, n.$$

The element  $z$  spans the one-dimensional center of  $\mathfrak{h}_n$ .

REMARK 5.1. The algebra  $\mathfrak{h}_n$  appears naturally in the context of symplectic geometry as a Poisson algebra of linear functions on the standard  $2n$ -dimensional symplectic space.

We say that  $M$  is equipped with a *Heisenberg structure* if one fixes a contact form  $\alpha$  on  $M$  and a  $\mathfrak{h}_n$ -action spanned by  $2n + 1$  vector fields  $\{A_1, \dots, A_n, B_1, \dots, B_n, Z\}$ , such that the  $2n$  vector fields  $A_i, B_j$  are independent at any point and tangent to the contact structure:

$$i_{A_i} \alpha = i_{B_j} \alpha = 0$$

and  $[A_i, B_i] = Z$ , where  $Z$  is the Reeb field, while the other Lie brackets are zero.

## 5.2 EXAMPLE: THE LOCAL HEISENBERG STRUCTURE

The Darboux theorem states that locally contact manifolds are diffeomorphic to each other. An effective way to formulate this theorem is to say that in a neighborhood of any point of  $M$  there is a system of local coordinates  $(x_1, \dots, x_n, y_1, \dots, y_n, z)$  such that the contact structure  $\xi$  is given by the 1-form

$$\alpha = \sum_{i=1}^n \frac{x_i dy_i - y_i dx_i}{2} + dz.$$

These coordinates are called the Darboux coordinates.

PROPOSITION 5.2. *The vector fields*

$$(10) \quad A_i = \frac{\partial}{\partial x_i} + \frac{y_i}{2} \frac{\partial}{\partial z}, \quad B_i = -\frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z},$$

where  $i = 1, \dots, n$ , define a Heisenberg structure on  $\mathbf{R}^{2n+1}$ .

*Proof.* One readily checks that  $A_i, B_j$  are tangent and

$$[A_i, B_j] = \delta_{ij} Z$$

while other commutation relations are zero. The vector field  $Z$  is simply the Reeb field.  $\square$

There is a well-known formula for a contact vector field in the Darboux coordinates (see e.g. [1, 2, 4]). We will not use this formula since the expression in terms of the Heisenberg structure is much simpler.

### 5.3 CONTACT VECTOR FIELDS AND HEISENBERG STRUCTURE

Assume that  $M$  is equipped with an arbitrary Heisenberg structure. It turns out that every contact vector field can be expressed in terms of the basis of the  $\mathfrak{h}_n$ -action by a universal formula.

**PROPOSITION 5.3.** *Given an arbitrary Heisenberg structure on  $M$ , a contact vector field with a contact Hamiltonian  $H$  is given by the formula*

$$(11) \quad X_H = HZ - \sum_{i=1}^n (A_i(H)B_i - B_i(H)A_i) .$$

*Proof.* Let us first check that the vector field (11) is, indeed, contact. If  $X$  is as on the right hand side of (11), then the Lie derivative  $L_X\alpha := (d \circ i_X + i_X \circ d)\alpha$  is given by

$$L_X\alpha = dH - \sum_{i=1}^n (A_i(H)i_{B_i} - B_i(H)i_{A_i})d\alpha .$$

To show that the 1-form  $L_X\alpha$  is proportional to  $\alpha$ , it suffices to check that

$$i_{A_i}(L_X\alpha) = i_{B_j}(L_X\alpha) = 0 \quad \text{for all } i, j = 1, \dots, n .$$

The first relation is a consequence of the formulæ  $i_{A_i}(dH) = A_i(H)$  together with

$$(12) \quad \begin{aligned} i_{A_i}i_{B_j}d\alpha &= i_{A_i}(L_{B_j}\alpha) = i_{[A_i, B_j]}\alpha = \delta_{ij}i_Z\alpha = \delta_{ij} , \\ i_{A_i}i_{A_j}d\alpha &= i_{B_i}i_{B_j}d\alpha = 0 . \end{aligned}$$

The second one follows from the similar relations for  $i_{B_j}$ .

Secondly, observe that, if  $X$  is as in (11), then  $i_X\alpha = H$ . This means that the contact Hamiltonian of the contact vector field (11) is precisely  $H$ .  $\square$

Note that a formula similar to (11) was used in [4] to define a contact structure.

### 5.4 THE ACTION OF $\text{CVect}(M)$ ON $\text{TVect}(M)$

Since  $2n$  vector fields  $A_i$  and  $B_j$  are linearly independent at any point, they form a basis of  $\text{TVect}(M)$  over  $C^\infty(M)$ . Therefore, an arbitrary tangent vector field  $X$  has a unique decomposition

$$(13) \quad X = \sum_{i=1}^n (F_i A_i + G_i B_i) ,$$

where  $(F_i, G_j)$  in a  $2n$ -tuple of smooth functions on  $M$ . The space  $\text{TVect}(M)$  is now identified with the direct sum

$$\text{TVect}(M) \cong \underbrace{C^\infty(M) \oplus \cdots \oplus C^\infty(M)}_{2n \text{ times}}.$$

Let us calculate explicitly the action of  $\text{CVect}(M)$  on  $\text{TVect}(M)$ .

**PROPOSITION 5.4.** *The action of  $\text{CVect}(M)$  on  $\text{TVect}(M)$  is given by the first-order  $(2n \times 2n)$ -matrix differential operator*

$$(14) \quad X_H \begin{pmatrix} F \\ G \end{pmatrix} = \left( X_H \cdot \mathbf{1} - \begin{pmatrix} AB(H) & BB(H) \\ -AA(H) & -BA(H) \end{pmatrix} \right) \begin{pmatrix} F \\ G \end{pmatrix},$$

where  $F$  and  $G$  are  $n$ -vector functions,  $\mathbf{1}$  is the unit  $(2n \times 2n)$ -matrix,  $AA(H)$ ,  $AB(H)$ ,  $BA(H)$  and  $BB(H)$  are  $(n \times n)$ -matrices, namely

$$AA(H)_{ij} = A_i A_j(H),$$

and the three other expressions are similar.

*Proof.* Straightforward from (11) and (13).  $\square$

**PROPOSITION 5.5.** *The bilinear map (7) has the following explicit expression:*

$$H_{X, \tilde{X}} = \sum_{i=1}^n \begin{vmatrix} F_i & \tilde{F}_i \\ G_i & \tilde{G}_i \end{vmatrix},$$

where  $X = \sum_{i=1}^n (F_i A_i + G_i B_i)$ , and  $\tilde{X} = \sum_{j=1}^n (\tilde{F}_j A_j + \tilde{G}_j B_j)$ .

*Proof.* This follows from definition (7) and formula (12).  $\square$

Note that formula (14) implies that  $H_{X, \tilde{X}}$  transforms as a contact Hamiltonian according to (3) since the partial traces of the  $(2n \times 2n)$ -matrix in (14) are  $A_i B_i(H) - B_i A_i(H) = Z(H)$ .

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V. Ovsienko

CNRS, Institut Camille Jordan  
Université Claude Bernard Lyon 1  
21, avenue Claude Bernard  
F-69622 Villeurbanne Cedex  
France  
*e-mail*: ovsienko@math.univ-lyon1.fr



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