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# ON THE GEOMETRY OF HOLONOMY SYSTEMS

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ABSTRACT. We give a geometric proof of the theorem of Simons on holonomy systems, which implies the Berger holonomy theorem. The proof is based on submanifold geometry and, in particular, on normal holonomy. This is part of a project of geometrization of representation theory. We also give some applications of Simons' result to submanifold geometry.

## 1. INTRODUCTION

The holonomy group $\Phi$, associated to a connected Riemannian manifold $M$ at a given point $p$, is a Lie group of orthogonal transformations of the tangent space $T_pM$. It is obtained by parallel transport of tangent vectors along arbitrary loops based at $p$. The holonomy groups at different points of a connected manifold $M$ are isomorphic. In fact, parallel transport along any curve joining $p$ and $q$ conjugates the respective holonomy groups. The holonomy group measures the deviation of the space from being globally flat, in which case $\Phi$ is trivial. The connected component of the identity of $\Phi$ is called the restricted holonomy group. It coincides with the holonomy group of the universal cover of $M$. The holonomy group of a small connected open neighbourhood $U$ of $p \in M$ is called the local holonomy group at $p$. Namely, if $V \subset U$ is an open neighbourhood of $p$ then the holonomy groups of $U$ and $V$ coincide. Holonomy groups play a central role in Riemannian geometry. The reducibility of the $\Phi$-action on the tangent space $T_pM$ implies, via the decomposition theorem of de Rham, the local product decomposition of $M$. In fact, a $\Phi$-irreducible subspace $V$ of $T_pM$ gives rise to a parallel distribution on $M$. The Riemannian manifold $M$ is called irreducible if the

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restricted holonomy group acts irreducibly (a general reference for holonomy can be found in chapter 9 of [Bes] or in [Sa], [J]).

A well known theorem of Ambrose and Singer relates the Lie algebra $\mathcal{G}$ of $\Phi$ with the curvature tensor $R$. Namely, $\mathcal{G}$ coincides with the linear span of the set $\{\tau_c(R_{X_q,Y_q}) : q \in M,\ X_q,\ Y_q \in T_qM\}$ where $c$ is an arbitrary curve from $q$ to $p$ and $\tau$ denotes the parallel transport.

The following result is a consequence of Berger's classification [Be] of the possible holonomy groups of non-locally symmetric spaces.

BERGER HOLONOMY THEOREM. *If the holonomy group of an irreducible Riemannian manifold $M$ at $p$ is not transitive on the sphere of the tangent space $T_pM$, then $M$ is locally symmetric.*

Apart from the basic facts of the theory, this is the most important and beautiful general result in Riemannian geometry. By 'general' we mean that there are no further assumptions on the space such as curvature assumptions, compactness, etc.

The existence of parallel tensors or, more generally, parallel geometric structures on $M$ is equivalent to the existence of algebraic tensors or structures on the tangent space $T_pM$, which are invariant under the holonomy group $\Phi$. For locally symmetric spaces this question of existence is well understood. In this case the holonomy coincides with the isotropy. So, this question is reduced, by means of Berger's theorem, to the study of invariant algebraic structures under transitive groups, i.e. orthogonal Lie groups which are transitive on the sphere. Not all transitive groups arise as Riemannian holonomy. Transitive holonomies are listed below ([Sa, chap. 10] and [J]):

- $SO(n)$, *associated to generic $n$-dimensional manifolds.*
- $U(n)$, *associated to generic Kähler manifolds of real dimension* $2n$.
- $Sp(n).Sp(1)$, *quaternionic-Kähler manifolds,* $\dim = 4n$. *All are Einstein.*
- $Spin(9)$, *always symmetric, the Cayley plane or its dual,* $\dim = 16$ ([A], [BG]).
- $SU(n)$, *Calabi-Yau manifolds,* $\dim = 2n$. *Kähler and Ricci-flat.*
- $Sp(n)$, *hyper-Kähler manifolds,* $\dim = 4n$. *Kähler and Ricci-flat.*
- $G_2$, *the so-called $G_2$-manifolds,* $\dim = 7$. *Ricci-flat.*
- $Spin(7)$, *the so-called $Spin(7)$-manifolds,* $\dim = 8$. *Ricci-flat.*

For the first four types of transitive groups listed above, there are symmetric spaces, which must be of rank one, which have them as holonomy groups. These are the non-exceptional holonomies.

Other important and non-trivial applications of the Berger holonomy theorem are the rank rigidity results. The rank of a Riemannian manifold is the maximal $k$ such that every geodesic is contained in a $k$-flat, i.e. a $k$-dimensional totally geodesic and flat submanifold. In this direction one has Ballmann's proof of his celebrated result (independently proved by Burns and Spatzier [BS]): *a complete irreducible Riemannian manifold of non-positive sectional curvature, finite volume, and rank at least two must be locally symmetric.* The dual to this result, by Molina and the author [MO], also makes use of Berger's theorem. In this case flats are assumed to be compact and there is no assumption on the sectional curvatures. The conclusion is the same: local symmetry.

To obtain the list of possible non-transitive holonomies, Berger exploited the fact that the curvature tensor at a point takes values in the holonomy algebra. The proof is long and he also used the fact that the covariant derivative of the curvature tensor takes values in the holonomy algebra. Some years later, James Simons [Si] gave a classification-free proof of the Berger holonomy theorem. He defined the concept of *holonomy system.* This is a triple $[\mathbf{V}, R, G]$, where $\mathbf{V}$ is a Euclidean vector space, $G$ is a compact connected subgroup of the linear isometries of $\mathbf{V}$, and $R$ is an algebraic curvature tensor which takes values in the Lie algebra $\mathcal{G}$ of $G$ (i.e. $R_{u,v} \in \mathcal{G}$ for all $u, v \in \mathbf{V}$). Such a triple is called:

- *irreducible* if $G$ acts irreducibly on $\mathbf{V}$;
- *transitive* if $G$ acts transitively on the unit sphere of $\mathbf{V}$;
- *symmetric* if $g(R) = R$ for all $g \in G$, where $g(R)_{u,v} = g^{-1} R_{g.u, g.v}\, g$.

SIMONS' HOLONOMY THEOREM [Si]. *An irreducible Riemannian holonomy system which is not transitive must be symmetric.*

The above result implies Berger's holonomy theorem, as shown by Simons (see Section 3).

Except from the first general part, the proof given by Simons is algebraic and involved. He makes use of case by case arguments combined with double inductions. It is difficult to go through all the details of the proof. The author [O3] recently succeeded in giving a geometric proof of Berger's theorem. This proof completely avoids the use of holonomy systems. It makes use of submanifold geometry and gives a link between Riemannian holonomy groups and normal holonomy groups (i.e. holonomy groups of the normal connection of submanifolds).

The aim of this article is to give a conceptual proof of the Simons holonomy theorem, based on the submanifold geometry of orbits. In particular this gives an alternative proof of the Berger holonomy theorem, shorter than that given in [O3]. It should be remarked that the methods in this article, though motivated by [O3], are not an immediate consequence of those in this reference and depend also on other observations. This article illustrates the importance of geometric tools (such as normal holonomy groups) for obtaining algebraic results concerning orthogonal groups. This is part of a project of geometrization of representation theory (results in this direction are, for instance, those cited as 68, 69, 81, 91, 175, 176 in [BCO]). We believe that Simons' holonomy theorem is a beautiful and strong algebraic result, which could have, in Simons' words, many other interesting applications. We give such an application to submanifold geometry in Section 4.

## 2. Preliminaries and basic facts

The general reference for this section is [BCO] (see also the survey article [CDO]).

Let $\mathbf{V}$ be a Euclidean vector space. A tensor $R\colon \mathbf{V} \times \mathbf{V} \to \mathfrak{so}(\mathbf{V})$ is called an *algebraic curvature tensor* if it satisfies the algebraic identities of a Riemannian curvature tensor at a point. Namely,

(a) $\langle R_{u,v}z, w\rangle = -\langle R_{v,u}z, w\rangle$ ;

(b) $\langle R_{u,v}z, w\rangle = -\langle R_{u,v}w, z\rangle$ ;

(c) $\langle R_{u,v}z, w\rangle = \langle R_{z,w}u, v\rangle$ ;

(d) $R_{u,v}z + R_{v,z}u + R_{z,u}v = 0$ (first Bianchi identity).

It is well known that (c) is a consequence of the other three identities.

Let $M^n$ be a submanifold of a space of constant curvature $Q^{n+k}$ and let $\nu M = \{(T_pM)^\perp : p \in M\}$ denote the normal space of $M$. Let $\nabla^\perp$ be the normal connection in $\nu(M)$ and let

$$R^\perp_{u,v}\xi = \nabla^\perp_u \nabla^\perp_v \xi - \nabla^\perp_v \nabla^\perp_u \xi - \nabla^\perp_{[u,v]}\xi$$

be the normal curvature tensor. Let $A$ denote the shape operator of $M$. Then one has the following well known Ricci identity:

$$\langle R^\perp_{u,v}\xi, \eta\rangle = \langle [A_\xi, A_\eta]u, v\rangle .$$

Since the above expression is skew-symmetric both in tangent and normal directions, one has that the normal curvature tensor at $p$ induces the linear map

$$\widetilde{R}_p^{\perp}: \Lambda^2(T_pM) \to \Lambda^2(\nu_pM).$$

Namely, $\langle \widetilde{R}_p^{\perp}(u \wedge v), \xi \wedge \eta \rangle = \langle R_{u,v}^{\perp}\xi, \eta \rangle$. Let now $(\widetilde{R}_p^{\perp})^t: \Lambda^2(\nu_pM) \to \Lambda^2(T_pM)$ be the transpose homomorphism and let

$$\widetilde{\mathcal{R}}_p^{\perp} := \widetilde{R}_p^{\perp} \circ (\widetilde{R}_p^{\perp})^t: \Lambda^2(\nu_pM) \to \Lambda^2(\nu_pM).$$

The image of $\widetilde{\mathcal{R}}_p^{\perp}$ coincides with the image of $\widetilde{R}_p^{\perp}$. This tensor induces the so-called *adapted normal curvature tensor* $\mathcal{R}^{\perp}: (\nu M)^3 \to \nu M$, given by $\langle \mathcal{R}_{\xi,\eta}^{\perp}\mu, \psi \rangle = \langle \widetilde{\mathcal{R}}^{\perp}(\xi \wedge \eta), \mu \wedge \psi \rangle$. From the Ricci identity one has:

$$\langle \mathcal{R}_{\xi,\eta}^{\perp}\mu, \psi \rangle = \langle [A_\xi, A_\eta], [A_\mu, A_\psi] \rangle = -\mathrm{trace}([A_\xi, A_\eta].[A_\mu, A_\psi]).$$

This formula implies that $\mathcal{R}^{\perp}$ is an algebraic curvature tensor of the normal space, which takes values in the normal holonomy algebra (see Chapter IV of [BCO]). Moreover, the scalar curvature of $\mathcal{R}^{\perp}$ is non-zero, unless $\mathcal{R}^{\perp} = 0$. By taking the average of $\mathcal{R}^{\perp}$ over the restricted normal holonomy group and applying Cartan's construction of symmetric spaces (see Section 3) one gets that the restricted holonomy group of the normal connection, in the orthogonal complement of its fixed point set, acts as an $s$-representation, i.e. the isotropy representation of a semisimple symmetric space [O1].

NORMAL HOLONOMY THEOREM. *Let $M$ be a submanifold of a space of constant curvature and let $p \in M$. Then the restricted normal holonomy group of $M$ at $p$ acts, in the orthogonal complement of its fixed point set, as an $s$-representation.*

The following result is well known (for a proof see [BCO. p. 192]):

LEMMA 2.1. *If the subgroup $H$ of $SO(m)$ acts on $\mathbf{R}^m$ as an $s$-representation then $N_o(H) = H$, where $N_o(H)$ denotes the identity component of the normalizer of $H$ in $SO(m)$.*

Let us recall that a submanifold of Euclidean space is called *full* if it is not contained in any proper affine subspace. A submanifold of the sphere is said to be full if it is full when regarded as a Euclidean submanifold.

The next proposition is not easy to obtain. It follows from Proposition 4.7 of [O2] or Theorem 1.1 in [DO]. It is a crucial fact for proving the rank rigidity theorem for homogeneous Euclidean submanifolds: *An irreducible full homogeneous submanifold of Euclidean space of rank at least* 2*, which is not a curve, is an orbit of an $s$-representation* (see Theorem A in [O2], Theorem 1.2 in [DO] and Chapter VI of [BCO]). The rank of a Euclidean submanifold is the maximal number of linearly independent, locally defined, parallel normal fields. Observe that the rank of a submanifold which is contained in a sphere is at least 1, since the position vector is a parallel normal field. Moreover, a homogeneous Euclidean submanifold with rank at least 1 must always be contained in a sphere (see Corollary 6.1.8 in [BCO]).

PROPOSITION 2.2. *Let $M = G.v$ be a full homogeneous submanifold of $\mathbf{R}^m$, where $G$ is a connected Lie subgroup of $SO(m)$. Assume that the submanifold $M$ has no one-dimensional extrinsic factors. Then any locally defined parallel normal field of $M$ is $G$-invariant.*

For the proof of Simons' theorem we will need to make some observations. Let $G$ be a Lie subgroup of the isometries of a Euclidean vector space $\mathbf{V}$, with Lie algebra $\mathcal{G}$. For $X \in \mathcal{G}$, let $\widetilde{X}$ be the Euclidean Killing field defined by $q \mapsto \frac{d}{dt}_{|0} \operatorname{Exp}(tX).q = X.q$. If $\mathbf{W}$ is an affine subspace of $\mathbf{V}$ then the orthogonal projection into $\mathbf{W}$ of the restriction $\widetilde{X}|_{\mathbf{W}}$ is a Killing field of $\mathbf{W}$. Such a Killing field of $\mathbf{W}$ will simply be called *the projection into* $\mathbf{W}$ of the Killing field $\widetilde{X}$.

LEMMA 2.3 ([OS], Lemma 3.1). *Let $v \in \mathbf{V}$ and let $X \in \mathcal{G}$. Then the projection of the Killing field $\widetilde{X}$, induced by $X$, into the affine normal space $v + \nu_v(G.v)$ belongs to the Lie algebra of the normalizer, in the orthogonal group of the normal space, of the normal holonomy group $\Phi^\perp$ of $G.v$ at $v$.*

*Proof.* Let $\tau_t^\perp$ be the $\nabla^\perp$-parallel transport in $G.v$ along the curve $\operatorname{Exp}(tX).v$. Then

$$(*) \qquad (\tau_t^\perp)^{-1} \circ \operatorname{Exp}(tX)_{*|\nu_v(G.v)}$$

must lie in the connected component of the normalizer of $\Phi^\perp$, for all $t \in \mathbf{R}$, since any extrinsic isometry of $G.v$ must map normal holonomy groups into normal holonomy groups. Therefore, differentiating at $t = 0$ in $(*)$, one obtains that $\bar{X} := \frac{D^\perp}{dt} \operatorname{Exp}(tX)_{*|\nu_v(G.v)}$ lies in the Lie algebra of the normalizer of $\Phi^\perp$.

But $\bar{X}.w = (\widetilde{X}.w)^{\perp}$, where $(\ )^{\perp}$ denotes the projection into $\nu_v(G.v)$. This implies the lemma. $\square$

If $G$ acts irreducibly on $\mathbf{V}$ then any orbit $G.v$, $v \neq 0$, must be a full and irreducible submanifold of $\mathbf{V}$. Moreover, the dimension of any orbit $G.v$ is at least 2, unless $\dim(\mathbf{V}) = 2$. In fact, if $\dim(G.v) = 1$, then any $X$ in the isotropy algebra of $G$ at $v$ must be zero, since it induces a trivial isometry on the curve $G.v$ which spans the ambient space. So, $\dim(G) = 1$ and therefore $\dim(\mathbf{V}) = 2$, since $G$ acts irreducibly. The following result is implicit in [OS].

PROPOSITION 2.4. *Let $G$ be Lie subgroup of linear isometries of $\mathbf{V}$. Let $0 \neq v \in \mathbf{V}$ and let $X \in \mathcal{G}$. Assume furthermore that $G$ acts irreducibly on $\mathbf{V}$. Then the projection of the Killing field $\widetilde{X} := q \mapsto X.q$ into the affine normal space $v + \nu_v(G.v)$ belongs to the Lie algebra of the normal holonomy group $\Phi^{\perp}$ of $G.v$ at $v$.*

*Proof.* If $\dim(\mathbf{V}) = 2$, the proof is trivial, since non-trivial orbits are extrinsic circles. So, let us assume that $\dim(\mathbf{V})$ is at least 3. Then, as previously observed, $\dim(G.v) \geq 2$ and $G.v$ is a full and irreducible Euclidean submanifold. Therefore we can apply Proposition 2.2 to conclude that any parallel normal field to $G.v$ must be $G$-invariant. Let us decompose the normal space $\nu_v(G.v) = \nu_0 \oplus \nu_0^{\perp}$, where $\nu_0$ is the fixed set of the restricted normal holonomy group $\Phi_*^{\perp}$ of $G.v$ at $v$. Let $X \in \mathcal{G}$ and let $\bar{X}$ be the projection of $\widetilde{X}$ into the normal space $\nu_v(G.v)$. Then $\bar{X}$ is trivial when restricted to $\nu_0$. In fact, if $\xi \in \nu_0$, $\mathrm{Exp}(tX).\xi = \tilde{\xi}(\mathrm{Exp}(tX).v)$, where $\tilde{\xi}$ is the extension of $\xi$ to a parallel normal field. Differentiating the last equality at $t = 0$ one obtains that $X.\xi$ must be tangent to $G.v$. So, $\bar{X}.\xi = 0$.

By the normal holonomy theorem, the group $\Phi_*^{\perp}$, which has the same Lie algebra as $\Phi^{\perp}$, acts on $\nu_0^{\perp}$ as an $s$-representation. The fact that $\bar{X}$ does belong to the holonomy algebra now follows directly from Lemma 2.3 and Lemma 2.1. $\square$

Let $G$ be a Lie subgroup of the orthogonal group $SO(n)$ and let $v \in \mathbf{R}^n$ be such that the orbit $G.v$ is of maximal dimension (this is always true for $v$ in an open and dense subset of $\mathbf{R}^n$). It is standard and well known that the connected component $(G_v)_0$ of the isotropy subgroup of $G$ at $v$ acts trivially on the normal space $\nu_v(G.v)$. So any $\xi \in \nu_v(G.v)$ defines, locally, a $G$-invariant normal vector field $\tilde{\xi}$ with $\tilde{\xi}(v) = \xi$.

We need the following result (see Lemma 2.2 and Remark 2.1 of [O3]).

LEMMA 2.5 ([O3]). *Let $G$ be a Lie group of linear isometries of $\mathbf{R}^n$ which is not transitive on the sphere and let $v \in \mathbf{R}^n$ be such that the orbit $G.v$ is of maximal dimension. Then there exists $\xi \in \nu_v(G.v)$, not a multiple of $v$, such that the family of normal spaces $\nu_{\gamma(t)}(G.\gamma(t))$ spans $\mathbf{R}^n$, where $\gamma(t) = v + t\xi$, $t \in \mathbf{R}$ (in fact, such a $\xi$ is generic). Moreover, $v$ belongs to any element of this family of normal spaces.*

*Proof.* Choose $\xi \in \nu_v(G.v)$ which is not a multiple of the position vector $v$. We may assume that $\det(A_\xi) \neq 0$ (i.e. that all eigenvalues of $A_\xi$ are different from $0$). In fact, if $\det(A_\xi) = 0$, we can add to $\xi$ a small multiple of the position vector $v$, since its shape operator is $A_v = -Id$. Let $\gamma(t) = v + t\xi$ and let $\mathbf{V}$ be the orthogonal complement of the linear span of the family $\nu_{\gamma(t)}(G.\gamma(t))$, $t \in \mathbf{R}$. We want to show that $\mathbf{V} = \{0\}$.

By construction, $\mathbf{V}$ is contained in all tangent spaces $T_{\gamma(t)}(G.\gamma(t))$. Let $X$ belong to the Lie algebra $\mathcal{G}$ of $G$ and such that $X.v \in \mathbf{V}$. Let us denote by $J_\xi(t)$ the restriction to $\gamma$ of the Euclidean Killing field $\tilde{X}$. If $w := J'_\xi(0)$, then $J_\xi(t) = X.v + tw$. Since $J_\xi(t)$ is tangent to the orbit $G.\gamma(t)$, we obtain that $w \perp \nu_{\gamma(t)}(G.\gamma(t))$, for $t \neq 0$. But, for small $t$, $G.\gamma(t)$ has maximal dimension and so, the normal spaces to the associated orbits converge to $\nu_v(G.v)$. Then $w$ is also perpendicular to $\nu_v(G.v)$. Hence $w \in \mathbf{V}$. Since, as it is standard to check, $J'_\xi(0) = \nabla^\perp_{X.v}\tilde{\xi} - A_\xi(X.v)$ and $X.v$ is arbitrary in $\mathbf{V}$, we conclude that

$$\nabla^\perp_{\mathbf{V}}\tilde{\xi} = 0, \quad A_\xi(\mathbf{V}) \subset \mathbf{V}.$$

So, if $\mathbf{W} = \mathbf{V}^\perp \cap T_v(G.v)$, we also have that $A_\xi(\mathbf{W}) \subset \mathbf{W}$. Let now $Y \in \mathcal{G}$ be such that $Y.v \in \mathbf{W}$. Then the Jacobi field $\bar{J}_\xi(t)$ along $\gamma(t)$, induced by $Y$, has initial conditions $Y.v$, $\nabla^\perp_{Y.v}\tilde{\xi} - A_\xi(Y.v)$ both of which lie in $\mathbf{V}^\perp$. So, $\bar{J}_\xi(t) \perp \mathbf{V}$. Let now $X_1, \dots, X_k \in \mathcal{G}$ be such that $X_1.v, \dots, X_k.v$ is an orthonormal basis which diagonalizes the restriction to $\mathbf{V}$ of $A_\xi$. Then their associated Jacobi fields along $\gamma$ are $J^i_\xi(t) = (1 - t\lambda_i)X_i.v$, where $\lambda_i \neq 0$ is the eigenvalue of $A_\xi$ associated to $X_i.v$ $(i = 1, \dots, k)$. Let now $Z \in \mathcal{G}$ be arbitrary and write $Z = X + Y$, where $X$ is a linear combination of $X_1, \dots, X_k$ and $Y.v \in W$. From this we obtain that the Jacobi field, induced by $Z$ along $\gamma$, at $t = 1/\lambda_i$ is perpendicular to $X_i.v$. Since $Z$ is arbitrary we have that $X_i.v \in \nu_{\gamma(1/\lambda_i)}(G.\gamma(1/\lambda_i))$ which is a contradiction, unless $\mathbf{V} = \{0\}$. $\square$

## 3. IRREDUCIBLE HOLONOMY SYSTEMS

The following result relates holonomy systems to normal holonomy of orbits.

PROPOSITION 3.1. *Let* $[\mathbf{V}, R, G]$ *be a holonomy system. Then*

(i) *The normal space* $\nu_v(G\,.\,v)$ *is left invariant by* $R$ *for all* $v \in \mathbf{V}$, *i.e.* $R_{\nu_v(G.v),\nu_v(G.v)}\nu_v(G\,.\,v) \subset \nu_v(G\,.\,v)$.

(ii) *The restriction* $R^v$ *of* $R$ *to* $\nu_v(G\,.\,v)$ *is left invariant by the normal holonomy group* $\Phi^\perp$ *of* $G\,.\,v$ *at* $v$, *i.e.* $g(R^v) = R^v$ *for all* $g \in \Phi^\perp$.

*Proof.* Let $w, z \in \mathbf{V}$, $\xi \in \nu_v(G\,.\,v)$. Since $R_{w,z} \in \mathcal{G}$, $0 = \langle R_{w,z}v, \xi\rangle = \langle R_{v,\xi}w, z\rangle$. Hence $R_{v,\xi} = 0$. Let now $\eta$ also be in $\nu_v(G\,.\,v)$. Then, by the Bianchi identity, $R_{\xi,\eta}v = R_{v,\eta}\xi + R_{\xi,v}\eta = 0$. Thus, $R_{\xi,\eta}$ belongs to the isotropy subalgebra $\mathcal{G}_v$. But, $\mathcal{G}_v.\nu_v(G\,.\,v) \subset \nu_v(G\,.\,v)$. This proves part (i). (Observe that $R$ must also leave invariant any normal space $\nu_z(G\,.\,v)$, since $G\,.\,v = G\,.\,z$.)

Let $c(t)$ be a piece-wise smooth curve in $G\,.\,v$ with $c(0) = v$ and let $\tau_t^\perp$ denote the $\nabla^\perp$-parallel transport along $c|_{[0,t]}$. Let $\xi_i \in \nu_v(G\,.\,v)$ $(i = 1, \ldots, 4)$. First observe that $R$ being constant, the Euclidean derivative $\nabla^E R = 0$. Hence, if $\tilde{\xi}_i(t) = \tau_t^\perp(\xi_i)$,

$$\begin{aligned} 0 = \frac{d}{dt}\langle R_{\tilde{\xi}_1(t),\tilde{\xi}_2(t)}\tilde{\xi}_3(t), \tilde{\xi}_4(t)\rangle &- \langle R_{\frac{d}{dt}\tilde{\xi}_1(t),\tilde{\xi}_2(t)}\tilde{\xi}_3(t), \tilde{\xi}_4(t)\rangle \\ &- \langle R_{\tilde{\xi}_1(t),\frac{d}{dt}\tilde{\xi}_2(t)}\tilde{\xi}_3(t), \tilde{\xi}_4(t)\rangle - \langle R_{\tilde{\xi}_1(t),\tilde{\xi}_2(t)}\frac{d}{dt}\tilde{\xi}_3(t), \tilde{\xi}_4(t)\rangle \\ &- \langle R_{\tilde{\xi}_1(t),\tilde{\xi}_2(t)}\tilde{\xi}_3(t), \frac{d}{dt}\tilde{\xi}_4(t)\rangle\,. \end{aligned}$$

But $\frac{d}{dt}\tilde{\xi}_i(t)$ belongs to the tangent space $T_{c(t)}(G\,.\,v)$ $(i = 1, \ldots, 4)$. Then, using part (i), one obtains that

$$\frac{d}{dt}\langle R_{\tilde{\xi}_1(t),\tilde{\xi}_2(t)}\tilde{\xi}_3(t), \tilde{\xi}_4(t)\rangle = 0\,,$$

which implies part (ii) (this is in fact a special case of Lemma 1 of [E]). □

The proof of the following lemma is straightforward.

LEMMA 3.2. *Let $[\mathbf{V}, R, G]$ be a holonomy system, $X \in \mathcal{G}$, and let $\mathbf{W}$ be a linear subspace of $\mathbf{V}$ which is left invariant by both $R$ and $X.R$ (the restrictions to $\mathbf{W}$ are denoted by $\bar{R}$ and $\overline{X.R}$, respectively). Let $\bar{X}$ be the projection to $\mathbf{W}$, of the restriction to $\mathbf{W}$, of the Killing field of $\mathbf{V}$ induced by $X$. Then $\overline{X.R} = \bar{X}.\bar{R}$*

*Proof of Simons' theorem.* Let $v$ be such that $G.v$ has maximal dimension. Let $\gamma(t) = v + t\xi$ be given by Lemma 2.5. Let, for $X \in \mathcal{G}$, $\bar{X}^t$ be the projection of $\widetilde{X}$ to the normal space $\nu_{\gamma(t)}(G.\gamma(t))$. By Proposition 3.1, this family of normal spaces must be invariant under $R$. Denote by $\bar{R}^t$ the restriction of $R$ to $\nu_{\gamma(t)}(G.\gamma(t))$. Observe that $[\mathbf{V}, X.R, G]$ is also a holonomy system and hence we can define $\overline{X.R}^t$, to be the restriction of $X.R$ to $\nu_{\gamma(t)}(G.\gamma(t))$. By Proposition 2.4 and Proposition 3.1 (ii) one has:

$$\bar{X}^t.\bar{R}^t = 0.$$

Hence, by Lemma 3.2, one has that $\overline{X.R}^t = 0$. Then, by Lemma 2.5, one has, in particular, that the Jacobi operator $J_v^X := (X.R)_{.,v}v$ is the null endomorphism of $\mathbf{V}$. Since vectors, whose $G$-orbits has maximal dimension, are dense we conclude that all Jacobi operators $J_z^X$ are null. Hence $X.R$ has zero sectional curvatures. Therefore $X.R = 0$. Since $X \in \mathcal{G}$ is arbitrary we conclude that $[\mathbf{V}, R, G]$ is symmetric. □

In order to prove the Berger holonomy theorem we give a variation of Simons' arguments. We first observe that any irreducible symmetric holonomy system $[\mathbf{V}, R, K]$, $R \neq 0$, determines an irreducible simply connected symmetric space with curvature $\frac{1}{4}R$. In fact,

$$\mathcal{G} = \mathfrak{k} \oplus \mathfrak{p}$$

is an irreducible orthogonal involutive Lie algebra, where $\mathfrak{k}$ is the Lie algebra of $K$, $\mathfrak{p} = \mathbf{V}$ and the bracket is given by:

a) $[X, Y] = X.Y - Y.X$, if $X, Y \in \mathfrak{k}$;

b) $[X, v] = -[v, X] = X.v$, if $X \in \mathfrak{k}$, $v \in \mathfrak{p}$;

c) $[v, w] = R_{v,w}$, if $v, w \in \mathfrak{p}$.

This irreducible orthogonal involutive Lie algebra determines a symmetric space with isotropy $K$ and curvature tensor $\frac{1}{4}R$. This is known as *Cartan's construction* of a symmetric space. Observe that the scalar curvature of an irreducible symmetric space is different from zero.

LEMMA 3.3. *Let* $[\mathbf{V}, R, G]$ *and* $[\mathbf{V}, R', G]$ *be two irreducible symmetric holonomy systems, where* $R \neq 0$. *Then* $R'$ *is a scalar multiple of* $R$.

*Proof.* We may assume that $R' \neq 0$. We have, by applying Cartan's construction, that $R$ and $R'$ have both non-zero scalar curvature (since irreducible symmetric spaces have this property). So, there is a real number $\mu$ such that $\bar{R} = R - \mu R'$ has zero scalar curvature. The holonomy system $[\mathbf{V}, \bar{R}, G]$ is also irreducible and symmetric. Thus $\bar{R} = 0$. □

LEMMA 3.4. *Let* $R$ *be an algebraic curvature tensor on* $\mathbf{V}$ *and let* $\lambda$ *be a linear function on* $\mathbf{V}$. *Assume that the tensor* $\lambda \otimes R$ *satisfies the second Bianchi identity, i.e.,* $\lambda(z)R_{u,v} + \lambda(u)R_{v,z} + \lambda(v)R_{z,u} = 0$. *Then* $\lambda = 0$ *or* $R = 0$.

*Proof.* Assume that $\lambda \neq 0$ Let $h \in \mathbf{V}$ be such that $\langle h, \cdot \rangle = \lambda$. Let $\mathbf{W}$ be the orthogonal complement of the linear span of $h$. One has that $R_{v,w} = 0$ for all $v, w \in \mathbf{W}$ (just by applying the second Bianchi identity to $\lambda(h)R_{u.v}$). Let $e_1 = \frac{1}{\|h\|}h$ and complete it to an orthonormal basis $e_1, \dots, e_n$ of $\mathbf{V}$. Let $R^l_{i,j,k} := \langle R_{e_i,e_j}e_k, e_l \rangle$. Observe that $R^l_{i,j,k} = 0$ if $i,j \neq 1$. The same equality is also true if $k, l \neq 1$, since $R^l_{i,j,k} = R^j_{k,l,i}$. Then the only interesting coefficients to compute are $R^1_{1\,j,k} = R^1_{1,k,j}$, $j, k \neq 1$. Therefore one has, for all $i,j,k,l = 1, \dots, n$, that $R^l_{i,j,k} = R^l_{i,k,j}$. But this implies, as is well known, that $R^l_{i,j,k} = 0$. In fact, interchanging the indices $i$ and $j$ and then $k$ and $i$, one obtains that $R^l_{i,j,k} = -R^l_{j,k,i}$. Iterating this procedure three times one gets that $R^l_{i,j,k} = -R^l_{i,j,k}$, for all $i,j,k,l = 1, \dots, n$, which implies that $R = 0$. □

*Proof of the Berger holonomy theorem.* If the holonomy group $\Phi$ does not act transitively on the tangent space $T_pM$ then, by Simons' holonomy theorem, $[T_pM, R, \Phi^*]$, is an irreducible symmetric holonomy system, where $R$ is the curvature tensor at $p$ and $\Phi^*$ is the connected component of $\Phi$. Observe that the fact that $M$ is locally irreducible says by definition that $\Phi^*$ acts irreducibly. Assume that $R_p \neq 0$. Let $\nabla_Z R$ be the covariant derivative of $R$ at $p$. The irreducible holonomy system $[T_pM, \nabla_Z R, \Phi^*]$ is also non-transitive and hence symmetric, for any $Z \in T_pM$. Then, by Lemma 3.3, $\nabla_Z R = \lambda(Z)R$, for some linear function $\lambda$. Since $\nabla R$ satisfies the second Bianchi identity one has, by Lemma 3.4, that $\nabla R = 0$. The same is true for any other point $q \in M$ where the curvature tensor is not zero, and at accumulation points of such points. But at those points of $M$ where $R$ is locally zero the covariant derivative of the curvature tensor also vanishes. So, $M$ is locally symmetric. □

## 4. Applications to submanifold geometry

Let $M^n$ be a full submanifold of $\mathbf{R}^{n+k}$, i.e. $M$ is not contained in a proper affine subspace of the ambient space. The first normal space $\nu_p^1(M)$ of $M$ at $p$ is the linear span of $\{\alpha(X,Y) : X, Y \in T_pM\} \subset \nu_p(M)$, where $\alpha$ is the second fundamental form. Equivalently, one has that $\nu_p^1(M) = \{\xi \in \nu_p(M) : A_\xi = 0\}^\perp$. Assume that the bundle $\nu^1(M)$ is parallel with respect to the normal connection and that $M$ is a full submanifold. Then the first normal bundle coincides with the normal bundle. In fact, by the Ricci identity, $(\nu^1(M))^\perp$ is a parallel and flat subbundle of $\nu(M)$. It $0 \neq \tilde{\xi}$ is a parallel normal field with initial condition in $(\nu^1(M))_p^\perp$, then $\tilde{\xi}: M \to \mathbf{R}^{n+k}$ must be constant, and so $M$ would lie in the hyperplane $p + (\tilde{\xi}(p))^\perp$.

The subspace $T_pM \oplus \nu_p^1(M)$ is the so-called second osculating space of $M$ at $p$. So, the condition $\nu_p^1(M) = \nu_p(M)$ means that the second osculating space coincides with the ambient space. This condition, which is in particular satisfied by tight or taut submanifolds, due to Kuiper [K] (see Theorem 2.5 in [GT]), gives a strong restriction on the codimension. In fact, $k \leq \frac{1}{2}n(n+1)$, since the map $\xi \mapsto A_\xi$, from the normal space at $p$ into the symmetric endomorphisms, is one to one. In the particular case that $M = K.v$ is a homogeneous submanifold one has that $n + k \leq m + \frac{1}{2}m(m+1)$, where $m = \dim(K)$. Representations such that the first normal space coincides with the normal space for any orbit, were studied in [CT, GT].

Normal holonomy has revealed to be an important tool for Euclidean submanifold geometry [BCO], with applications to Riemannian geometry (see [O3]). The following result is an application of Simons' holonomy theorem to normal holonomy. The local normal holonomy group is defined in a similar way to the local riemannian holonomy group.

THEOREM 4.1. *Let $M^n$ be a full submanifold either of Euclidean space or of the sphere, such that the local holonomy group at $p$ acts without non-zero fixed points, i.e. there are no locally defined non-trivial parallel normal fields around $p$. Assume, further, that no factor of the local normal holonomy is transitive on the sphere. Then there are points in $M$, arbitrarily close to $p$, where the first normal space coincides with the normal space. In particular,* $\mathrm{codim}(M) \leq \frac{1}{2}n(n+1)$.

*Proof.* If $M$ is a submanifold of the sphere we regard $M$ as a Euclidean submanifold. We may assume that $M$ is so small that the local normal holonomy group coincides with the normal holonomy group at $p$. Let us decompose orthogonally the normal bundle as $\nu M = \nu_0 \oplus \nu_1 \oplus \cdots \oplus \nu_r$ into the parallel subbundles associated to the irreducible factors of the normal holonomy group ($\nu_0$ is trivial if $M$ is Euclidean, and $\nu_0$ is the one-dimensional normal subbundle generated by the position vector if $M$ is contained in the sphere). The adapted normal curvature tensor $\mathcal{R}^{\perp}_{\xi,\eta}$ leaves this decomposition invariant, for any normal fields $\xi, \eta$. Moreover, $\mathcal{R}^{\perp} = \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^r$, where $\mathcal{R}^i$ is the restriction of $\mathcal{R}$ to $\nu_i$ and $\mathcal{R}^0 = 0$. Observe that $\mathcal{R}^i \neq 0$, for each $i \neq 0$ (otherwise, $\nu_i$ would be flat). It is standard to show that there is a point $q \in M$, arbitrarily close to $p$, such that $\mathcal{R}^i(q) \neq 0$, for all $i = 1, \ldots, r$. One has that $[(\nu_1)_q, \mathcal{R}^i(q), \Phi_i]$ is an irreducible non-transitive holonomy system, where $\Phi_i$ is the restriction to $(\nu_i)_q$ of the normal holonomy group at $q$ (for each $i \neq 0$). So, by Simons' Holonomy Theorem, this holonomy system must be symmetric. In particular, $\mathcal{R}^i(q)$ is the curvature tensor of an irreducible symmetric space. Then it is non-degenerate (i.e., if for some $\xi \in (\nu_i)_q$, $\mathcal{R}^i(q)_{\xi,\eta} = 0$ for all $\eta \in (\nu_i)_q$, then $\xi = 0$). Then the degeneracy of $\mathcal{R}^{\perp}$ is $\nu_0$. If $\xi \in \nu_q M$ is perpendicular to the first normal space at $q$, then the shape operator $A_\xi$ is zero. Then $\xi$ lies in the degeneracy of $\mathcal{R}^{\perp}$ (see section 2). Then $\xi \in (\nu_0)_q$. So, $\xi = 0$ if $M$ is not contained in a sphere. If $M$ is contained in a sphere then $A_\xi$ is a non-trivial multiple of the identity, unless $\xi = 0$, since $\nu_0$ is generated by the position vector. So in this case also, $A_\xi = 0$ implies $\xi = 0$. □

COROLLARY 4.2. *Let $M^n$, $n \geq 2$, be a homogeneous irreducible full submanifold of Euclidean space such that the normal holonomy group, in each irreducible factor, acts non-transitively on the sphere. Then the first normal space of $M$ coincides with the normal space.*

*Proof.* If rank$(M) = 0$ then we are done by the above theorem. If rank$(M) = 1$, then $M$ is contained in a sphere (see Corollary 6.1.8 of [BCO]), and the normal holonomy group of $M$, regarded as a submanifold of the sphere, has no fixed set. So, by the above theorem we are also done. If rank$(M) \geq 2$ then by the rank rigidity theorem for submanifolds $M$ coincides with an orbit of an $s$-representation. It is well known, in this case, that the first normal space of $M$ coincides with the normal space. □

The above corollary could be useful to give some insight on the conjecture posed in [O2], which is a generalization of the rank rigidity theorem for homogeneous submanifolds: *an irreducible and full homogeneous submanifold of the sphere, different from a curve, such that the normal holonomy group is not transitive must be an orbit of an $s$-representation.*

This conjecture, which is true if $\dim(M) = 2$ (see [BCO, p. 198]), is actually equivalent to the following two conjectures taken together. It is not true for non-homogeneous submanifolds since the normal holonomy is invariant under conformal diffeomorphisms of the ambient space.

(a) *Let $M$ be a homogeneous irreducible and full submanifold of the sphere, different from a curve, which is not an orbit of an $s$-representation. Then the normal holonomy group acts irreducibly.*

(b) *Let $M$ be a homogeneous and full submanifold of the sphere such that the normal holonomy acts irreducibly and is non-transitive. Then $M$ is an orbit of an $s$-representation.*

Corollary 4.2 might be useful in the proof of part (b).

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