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ON PRODUCTS OF DISJOINT BLOCKS  
OF CONSECUTIVE INTEGERS

by Maciej ULAS

ABSTRACT. In this note we disprove the conjecture of Erdős and Graham which states that for fixed  $k \geq 2$  and  $l \geq 4$  the product of  $k$  disjoint blocks of  $l$  consecutive integers is a square only finitely many times. We give two infinite families of solutions for  $k = l = 4$ .

Let  $l \geq 2$  be an integer, and set  $f(x) = (x+1) \cdot \dots \cdot (x+l)$ . For a fixed integer  $k \geq 1$  we consider the diophantine equation

$$(1) \quad y^2 = \prod_{i=1}^k f(x_i),$$

with the condition

$$(2) \quad 0 < x_1 < \dots < x_k, \quad x_j + l \leq x_{j+1}, \quad j = 1, \dots, k-1.$$

Condition (2) ensures that the blocks  $f(x_1), \dots, f(x_k)$  of consecutive integers are disjoint.

Erdős and Selfridge [1] have proved a celebrated theorem implying that for  $k = 1$  and each  $l \geq 2$  the only solutions of (1) are  $x_1 = -1, \dots, -l$ ,  $y = 0$ . On the other hand, if there is more than one block we have

**THEOREM 1.** *For  $l \leq 3$  and each  $k \geq 2$ , equation (1) has infinitely many solutions satisfying condition (2).*

*Proof.* For  $l = 2$  and  $k = 2$  we take

$$x_1 = n - 1, \quad x_2 = 4n^2 + 4n - 1, \quad n \geq 2.$$

For  $l = 2$  and  $k = 3$  we take

$$x_1 = n - 1, \quad x_2 = 3n, \quad x_3 = \frac{9n(n+1)}{2} - 1, \quad n \geq 2.$$

Since each  $k \geq 2$  is of the form  $2a+3b$  with nonnegative  $a, b$ , we obtain our conclusion for  $l = 2$ .

For  $l = 3$  and  $k = 2$ , K.R.S. Sastry takes

$$x_1 = n - 1, \quad x_2 = 2n - 1,$$

where  $n, m \in \mathbf{N}$  satisfy the equation

$$(n + 2)(2n + 1) = m^2,$$

which has infinitely many solutions [3].

For  $l = 3$  and  $k = 3$  we take, as does Skałba in [4],

$$x_1 = F_{2u-1} - 2, \quad x_2 = F_{2u+1} - 2, \quad x_3 = F_{2u}^2 - 2, \quad u \geq 2,$$

where  $F_n$  is the  $n$ -th term of the Fibonacci sequence.

As in the case  $l = 2$ , we see that for each  $k \geq 2$  equation (1) has infinitely many solutions satisfying (2).  $\square$

In contrast to this result Erdős and Graham conjecture in ([2], p. 67) that if  $l \geq 4$ , then for each fixed  $k$ , equation (1) has only finitely many solutions satisfying (2). This conjecture is also mentioned in [3] as Problem D17.

In connection with this problem M. Skałba proved in [4] that if we allow  $k$  to vary, the above conjecture is not true.

We disprove the Erdős and Graham conjecture by establishing the following

**THEOREM 2.** *For  $k = l = 4$ , equation (1) has infinitely many solutions satisfying (2).*

*Proof.* Let  $n$  be an integer greater than 1. We put

$$x_1 = 4n - 1, \quad x_2 = 4n + 3, \quad x_3 = 4n^2 + 7n - 1, \quad x_4 = 8n^2 + 14n + 1.$$

Then

$$y^2 = f(x_1)f(x_2)f(x_3)f(x_4),$$

where

$$\begin{aligned} y &= 16n(n+1)(2n+1)(2n+3)(4n+1)(4n+3) \\ &\quad \times (4n+5)(4n+7)(4n^2+7n+1)(4n^2+7n+2). \end{aligned}$$

We can also take

$$x_1 = 4n, \quad x_2 = 4(n+1), \quad x_3 = 4n^2 + 9n + 1, \quad x_4 = 8n^2 + 18n + 5,$$

and then

$$y = 16(n+1)(n+2)(2n+1)(2n+3)(4n+1)(4n+3) \\ \times (4n+5)(4n+7)(4n^2+9n+3)(4n^2+9n+4). \quad \square$$

In connection with the above theorem it is natural to ask which is the smallest integer  $k$  such that equation (1) with  $l = 4$  has infinitely many solutions satisfying (2).

We looked for solutions with  $l = 4$  and  $k = 2$  or  $k = 3$  using a computer.

For  $k = 2$  we computed all solutions of (1) which satisfy  $x_1 + 3 < x_2 < 10^5$ . In this range we found only one solution,  $x_1 = 32$ ,  $x_2 = 1679$ .

For  $k = 3$  we computed all solutions of (1) which satisfy condition (2) and  $x_3 < 5 \cdot 10^3$ . The results are shown in the following table.

$x_1$	$x_2$	$x_3$	$x_1$	$x_2$	$x_3$
3	12	23	19	95	482
3	15	167	45	158	844
4	13	47	64	74	132
7	24	62	131	321	2207
7	38	285	186	208	421
9	244	1022	245	283	494
10	66	2207	368	404	898
11	30	152	491	549	1103
17	30	339	920	1063	1841
17	47	167			

Since each  $k \geq 6$  is of the form  $4s$ ,  $4s+2$ ,  $4s+3$  or  $4s+5$ , we obtain from the above computations and parametric solution in Theorem 2

**THEOREM 3.** *For  $l = 4$  and any fixed  $k \geq 6$ , equation (1) has infinitely many solutions satisfying (2).*

I firmly believe that in the remaining case  $k = 5$  there are infinitely many solutions. In the case  $l \geq 5$  the following conjecture seems to be true.

**CONJECTURE 4.** *For each  $l \geq 5$  there exists an integer  $n = n(l)$  such that for each  $k \geq n$  equation (1) has infinitely many solutions satisfying condition (2).*

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