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A FIXED-POINT THEOREM AND RELATIVE ASPHERICITY

by Max FORESTER and Colin ROURKE

ABSTRACT. We give short new geometric proofs of theorems of Bogley and Pride and of Serre. Both follow quickly from a global fixed-point theorem. The Bogley-Pride theorem concerns aspherical relative group presentations and was applied in [5] to the multivariable adjunction problem. Serre's theorem is a basic result concerning group cohomology and finite subgroups.

1. INTRODUCTION

Suppose that a group G acts on a set X. We say that G has a *global fixed point* if there is a point $x \in X$ which is the unique fixed point of each element $g \in G$. Let p be a prime. A space is said to be $\mathbf{Z}/p\mathbf{Z}$*-acyclic* if all reduced homology (or equivalently if all reduced cohomology) groups with $\mathbf{Z}/p\mathbf{Z}$-coefficients vanish. A space has *finite* $\mathbf{Z}/p\mathbf{Z}$*-homological dimension* if all but a finite number of these groups vanish. This note is concerned with the following theorem.

THEOREM 1.1 (Global Fixed-Point Theorem). *Suppose that a finite group G acts cellularly on a CW-complex Q freely away from the 0-skeleton. Suppose further that*

(a) *for each prime factor p of $|G|$, Q is $\mathbf{Z}/p\mathbf{Z}$-acyclic, and*

(b) *for each element g of G of prime order p the quotient $Q/\langle g\rangle$ has finite $\mathbf{Z}/p\mathbf{Z}$-homological dimension.*

Then G has a global fixed point.

The theorem is closely related to standard results proved using Smith theory or Tate cohomology. A standard result in Smith theory states that if a finite p-group acts simplicially on a $\mathbf{Z}/p\mathbf{Z}$-acyclic simplicial complex which is finite-dimensional, then the fixed-point set is $\mathbf{Z}/p\mathbf{Z}$-acyclic [2, Ch. III, Theorem 5.2]. In particular, if the fixed-point set is discrete then it consists of exactly one point. A special case of Theorem 1.1 then follows directly. (For a proof based on Tate cohomology, due to Swan [7], see [3, Ch. VII, Theorem 10.5].)

We shall give a direct elementary proof of Theorem 1.1.

Further we shall apply the result to provide short geometric proofs of a theorem of Bogley and Pride [1] about aspherical relative presentations and of Serre's Theorem [6] from which Bogley and Pride deduced their theorem. Serre's theorem is a basic result in group theory. A standard result states that groups of finite cohomological dimension are torsion-free. Serre's theorem is effectively a relative version of this: if some subgroups of a group carry all the high dimensional cohomology, then they contain all the torsion. See Section 4 for a precise statement. It has been applied in particular by Huebschmann [6] to small cancellation groups and by Bogley and Pride [1] to aspherical relative presentations. A special case of the Bogley-Pride theorem was applied in [5] to the multivariable adjuction problem, where a short proof of this special case was given. Our proofs here use similar ideas.

2. Proof of the Global Fixed-Point Theorem

The theorem follows immediately from Lemmas 2.1 and 2.2.

LEMMA 2.1. *Every non-trivial element $g \in G$ fixes a unique vertex of Q.*

Proof. Denote $Q/\langle g\rangle$ by T and let $f\colon Q \to T$ be the natural projection.

STEP 1. *If g has prime order p then g has at least one fixed point.*

Suppose that g has no fixed points. We will inductively construct $\mathbf{Z}/p\mathbf{Z}$-cycles c_i in T in all dimensions. Start with c_0 any vertex and let $b_0 = f^{-1}c_0$. Then b_0 consists of p points and hence is zero in $H_0(Q, \mathbf{Z}/p\mathbf{Z}) = \mathbf{Z}/p\mathbf{Z}$. So b_0 is the boundary of a 1-chain a_1 and we define $c_1 = f(a_1)$. Now suppose that c_i has been constructed. Let $b_i = f^{-1}c_i$ which is a $\mathbf{Z}/p\mathbf{Z}$-cycle in Q and which p-fold covers c_i. Since Q is acyclic,

b_i is the boundary of a $\mathbf{Z}/p\mathbf{Z}$-chain a_{i+1} say. Then $c_{i+1} = f(a_{i+1})$ is the next cycle.

We claim that all these cycles are non-zero in $\mathbf{Z}/p\mathbf{Z}$-homology. It then follows that T has infinite $\mathbf{Z}/p\mathbf{Z}$-homological dimension, contradicting the hypotheses. Hence g has a fixed point.

To see the cycles are all essential notice that the construction is natural and maps to a similar construction in the universal $\mathbf{Z}/p\mathbf{Z}$-bundle. We use Milnor's construction for the universal bundle, namely $E = \lim_i *_i P$ where P is a p point space, $*_i$ denotes the i-fold join $P * P * \cdots * P$ (that is, i join operations on $i+1$ copies of P), and the action is the join of the cyclic action on P. If we apply the construction to the i-th stage $*_i P \to R = *_i P / \mathbf{Z}/p\mathbf{Z}$ then the cycles b_j are the subsets $*_j P$ for $j \leq i$. So in this case b_i is the top (fundamental) cycle in $*_i P$ and is therefore non-zero. But if any of the cycles c_j, $j \leq i$ is zero, so are all subsequent ones and then b_i would be zero. It follows that c_j, $j \leq i$ are non-zero in R and hence, in the limit all the c_i are non-zero.

STEP 2. *If g has prime order p then g has at most one fixed point.*

Suppose g fixes at least two points (which must be vertices). Choose two x, y say. Let c_1 be an arc in T from $f(x)$ to $f(y)$ and let $b_1 = f^{-1}(c_1)$ (a $\mathbf{Z}/p\mathbf{Z}$-cycle). Then b_1 bounds a chain a_2 in Q. Let $c_2 = f(a_2)$, a $\mathbf{Z}/p\mathbf{Z}$-cycle in T. The construction now proceeds as in step 1.

We claim that as before all these cycles in T are non-zero in the $\mathbf{Z}/p\mathbf{Z}$-homology. It then follows that T has infinite $\mathbf{Z}/p\mathbf{Z}$-homological dimension, contradicting the hypotheses. Hence g has at most one fixed point.

To see this consider the universal bundle E as before. We map $Q \to \Sigma(E)$ (the suspension of E) by mapping x and y to the two suspension points and any other fixed points to either suspension point. Now for each fixed point a let A be its link in Q. By universality choose an equivariant map $A \to E$. Then a neighbourhood of a is mapped conically. Finally the map is extended to map the rest of Q to E by universality. Then since the construction of the classes c_i is again natural it maps to a similar construction in $\Sigma(E)$. But here we are constructing the suspensions of the classes constructed in step 1 which are all non-zero.

To finish the proof of the lemma, suppose g has order n and fixes no point of Q. If g^k fixes a point x for some $k > 1$, then g^k also fixes gx

(which is not x), and then a suitable power of g contradicts step 2. Hence $\langle g \rangle$ acts freely on Q and then a power of g contradicts step 1. Similarly if g has at least two fixed points, then a power of g contradicts step 2.

LEMMA 2.2. *If a finite group G acts on a set X in such a way that each non-trivial element fixes a unique point, then G has a global fixed point.*

Proof. Let x_g denote the unique fixed point of g. Note that $h(x_g) = x_{hgh^{-1}}$ and hence G acts on $\{x_g \mid g \in G - \{1\}\}$. So without loss we may assume that $X = \{x_g \mid g \in G - \{1\}\}$. If $|G| = 1$ the result is obvious so we may assume that $|G| > 1$.

Denote the stabilizer of $x \in X$ by G_x. Choose any $x \in X$ and let $\mathcal{O} \subset X$ be the orbit containing x, and let $n = |\mathcal{O}|$. By the orbit stabiliser theorem $|G| = n\,|G_x|$. Notice that if $y \in \mathcal{O}$ then G_x and G_y are conjugate and hence $|G_x| = |G_y|$. Notice also that the hypothesis of unique fixed points implies that if $x \neq y$ then $G_x - \{1\}$ and $G_y - \{1\}$ are disjoint.

Now define $S = \{g \in G - \{1\} \mid x_g \in \mathcal{O}\}$. Then S is the union over $y \in \mathcal{O}$ of the disjoint non-empty sets $G_y - \{1\}$ and we have $|S| = n(|G_x| - 1)$, which implies that $|S| > \frac{1}{2}(|G| - 1)$. Since $\mathcal{O}$ was an arbitrary orbit there is not room for another such set S and we must have $S = G - \{1\}$. Thus $|G| - 1 = n(|G_x| - 1)$ which implies $n = 1$ and $G_x = G$. This completes the proof of Lemma 2.2, and of Theorem 1.1.

REMARK 2.3. We do not need Q to be a CW-complex (or the action of G to be cellular) for the proof of the Global Fixed-Point Theorem to work but merely that the fixed points form a discrete set and the inclusion of the fixed points is a cofibration.

3. THE BOGLEY-PRIDE THEOREM

Let (L, K) be a relative 2-complex (a CW-pair such that $L - K$ is at most 2-dimensional). We say that (L, K) is *relatively aspherical* if the map

$$\pi_2(K \cup L^{(1)}, K) \to \pi_2(L, K)$$

is surjective. As shown in [4, 3.1-3.3], this occurs if and only if

(a) $\pi_1(K) \to \pi_1(L)$ is injective, and

(b) the inclusion-induced map $\mathbf{Z}\pi_1(L) \otimes_{\mathbf{Z}\pi_1(K)} \pi_2(K) \to \pi_2(L)$ is an isomorphism.

This is the natural topological notion of asphericity but it should be noted that it differs from the combinatorial notion used in [1]. The difference concerns the definition of irreducibility of diagrams representing elements of $\pi_2(L, K)$; see [4].

THEOREM 3.1 (Bogley-Pride [1]). *If (L, K) is relatively aspherical then every finite subgroup of $\pi_1(L)$ is contained in a unique conjugate of $\pi_1(K)$.*

Proof. By adding cells of dimension $\geqslant 3$ we can arrange that all the homotopy groups of K vanish in dimensions 2 and above. This does not change the fact that (L, K) is relatively aspherical. The easiest way to see this is to use the diagram interpretation used in [4]: relative asphericity means that there are no irreducible diagrams over $\pi_1(K)$ using the cells of $L - K$. This only depends on $\pi_1(K)$ and the form of the added relators and hence is unchanged by a change in the higher homotopy groups of K. After adding the new cells $\pi_2(L)$ is trivial.

Let $\widetilde{L}$ be the universal cover of L and $\widetilde{K}$ the preimage of K in $\widetilde{L}$. Let $\widehat{L}$ be the 2-complex obtained from $\widetilde{L}$ by collapsing each connected component of $\widetilde{K}$ to a vertex. Since each of these components is contractible, the map $\widetilde{L} \to \widehat{L}$ is a homotopy equivalence, and so $\pi_1(\widehat{L})$ and $\pi_2(\widehat{L})$ are trivial. Then since $\widehat{L}$ is 2-dimensional, it is contractible.

Note that the induced action of $\pi_1(L)$ on $\widehat{L}$ is free away from the 0-skeleton, and the vertices have stabilisers equal to the conjugates of $\pi_1(K)$ in $\pi_1(L)$. Hence it suffices to show that every finite subgroup of $\pi_1(L)$ has a global fixed point. But L is contractible and hence acyclic (for all coefficients) and further it is 2-dimensional. Therefore every quotient has dimension 2 (and hence has homological dimension 2 with all coefficients). So the Global Fixed-Point Theorem applies and every finite subgroup of $\pi_1(L)$ has a global fixed point as required.

4. SERRE'S THEOREM

We take the statement of Serre's Theorem from Huebschmann [6].

THEOREM 4.1 (Serre). *Let G be a group and $\{G_i\}_{i \in I}$ a family of subgroups such that for every $q \geq q_0$ the canonical map $H^q(G, M) \to \prod_i H^q(G_i, M)$ is an isomorphism for every G-module M. Then each finite subgroup F of G is contained uniquely in a conjugate of one of the G_i (and does not meet any other such conjugate).*

Proof. Let K be the disjoint union of the $K(G_i, 1)$ for $i \in I$ and form the open wedge K^+ (i.e. add an arc to an external basepoint for each component) and then construct L, a $K(G, 1)$, by attaching cells to K^+.

Let $\widetilde{L}$ be the universal cover of L and $\widetilde{K}$ the inverse image of K in $\widetilde{L}$ (which comprises a number of disjoint copies of universal covers of the $K(G_i, 1)$'s). Then form $\widehat{L}$ by squeezing each component of $\widetilde{K}$ to a point. Then, since we are squeezing contractible subcomplexes, $\widehat{L}$ is contractible and G acts freely off the 0-skeleton. Further the stabilisers of the vertices are the conjugates of the subgroups G_i, so we have to prove that each finite subgroup F of G has a global fixed point.

To do this we use Theorem 1.1 with $Q = \widehat{L}$. The space $\widehat{L}$ is contractible, so we have hypothesis (a). We have to check (b).

Now if H is a subgroup of G then $\widehat{L}/H$ is formed from a cover of L by squeezing components of the preimage of K. But the cohomology hypotheses lift to any cover (since they are given "for any G-module") so $\widehat{L}/H$ is formed by squeezing a subspace which carries all but finitely many of the cohomology groups and hence by excision it has finite (co)homological dimension. Thus we have hypothesis (b) of Theorem 1.1.

REMARK 4.2. The Global Fixed-Point Theorem is stronger than needed to prove the Bogley-Pride or Serre theorems. In these applications Q was contractible instead of just $\mathbf{Z}/p\mathbf{Z}$-acyclic for certain p and $Q/\langle g \rangle$ was either finite dimensional or had finite homological dimension with all coefficients.

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