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NEWTON-LIKE POLYNOMIALS OF LINKS

by Alexander STOIMENOW *)

ABSTRACT. We show that the coefficients of the Conway polynomial of special alternating links satisfy certain inequalities.

1. INTRODUCTION

A classical theorem of Newton (see Theorem 53 in [8]) states that if all the zeros of a polynomial P in $\mathbf{R}[x]$ are real, then $[P]_{i-1}[P]_{i+1} < [P]_i^2$ for any integer i with $\min\deg P \leq i \leq \max\deg P$. (Here $[P]_i$ is the coefficient of x^i in P, and $\min\deg P$, $\max\deg P$ are the minimal resp. maximal i with $[P]_i \neq 0$.) Call a polynomial satisfying these inequalities *Newton-like*; if $[P]_{i-1}[P]_{i+1} \leq [P]_i^2$ for any i, call P *weakly Newton-like*. In knot theory, Newton-like polynomials (not necessarily with all zeros real) seem to occur in special situations. Thus we advance the following conjecture:

CONJECTURE 1. (1) *The Alexander polynomial of an alternating link is weakly Newton-like, and* (2) *if ∇_L is the Conway polynomial of a positive link of $n(L)$ components, then $\nabla_L(\sqrt{z}) \cdot z^{(1-n(L))/2}$ is Newton-like.*

Here the Alexander polynomial Δ and Conway polynomial ∇ are the polynomial invariants for links defined in [1] and [2], taking values in $\mathbf{Z}[t, t^{-1}]$ and $\mathbf{Z}[z]$ resp. (The Alexander polynomial is defined only up to units in $\mathbf{Z}[t, t^{-1}]$; we will choose the normalization depending on how it is convenient in the context.)

There is strong empirical evidence for this conjecture. In particular, part (1) (resp. part (2)) is true for (knots with) alternating (resp. positive) knot diagrams

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of at most 16 crossings. Part (1) is a natural strengthening of Fox's "trapezoidal conjecture" [5], proved by Hartley [9] for rational knots, and later in a more generalized form by Murasugi [18]. In this paper, we shall prove part (2) for the intersection of positive and alternating links, the special alternating links. In fact, we prove

THEOREM 1. *If L is a special alternating link, then any zero of $\nabla_L(\sqrt{z})$ (is real and) lies in the interval $[-4,0]$, or equivalently, all zeros of Δ_K lie on the complex unit circle.*

As explained above, part (2) of Conjecture 1 for special alternating links follows immediately from Theorem 1 and Theorem 53 in [8].

To further motivate Theorem 1, let us mention that both special alternating links and zeros of the Conway (or Alexander) polynomial have been an object of study for some time. Such links occurred first notably in the work of Murasugi [20, 21] and later for example in [3] (as building blocks for homogeneous links) and [24]. They have a close relation to graphs, and so our work can be translated in a graph-theoretical context [23]. (See in particular part 12 of Theorem 3 therein[1]), which is a reformulation of the Newton property of ∇.) As for zeros of the Alexander polynomial, they have been studied over the years and have importance in particular for monodromy of fibered links [27], divisibility [22] and orderability [26] of knot groups, and statistical mechanical models of the Alexander polynomial [16]. Lehmer's question on the existence of a polynomial minimizing the Mahler measure is also studied in the context of Alexander polynomials of links (see for example [10, 30]). Empirical calculations led to the conjecture (as reported to me by Murasugi, proposed by Hoste) that if z is a root of the Alexander polynomial of an alternating knot, then $\Re e\ z > -1$. Theorem 1 also proves this conjecture for special alternating links (since for such links $z = -1$ is not a root of Δ).

A brief overview of the paper is as follows. §2 contains some preliminaries on polynomials and link diagrams. §3 explains the motivation for and main tools in the proof of Theorem 1, the Tristram-Levine signatures. Some number theoretic and analytic lemmas are prepared in §4. In particular Corollary 1 proves the case of Theorem 1 for knots and square-free (i.e., without double zeros) Alexander polynomials. (When no double zeros occur, the implicit function theorem easily does the required work.) Then in §5 the proof of the

[1]) The reference given in [23] to [36] for the proof is outdated; the proof is given below in this paper.

general case of Theorem 1 is given, using Corollary 1 and an algebraic approximation argument. The difficulty is that this approximation has to be carried out from within the set of square-free special alternating knot polynomials. We will require several tools from elementary number theory, analysis and algebra, for example a theorem of Hilbert on irreducibility of polynomials in 2 variables and the matrix formula for the discriminant of a polynomial. We obtain the following approximation result, which applies in a somewhat broader context:

PROPOSITION 1. *Let for a class $\mathcal{C}$ of links,*

$$\mathcal{Z}_{\mathcal{C}} = \{z \in \mathbf{C} \setminus \{0\} : \exists L \in \mathcal{C} : \Delta_L(z) = 0\},$$

and

$$\mathcal{Z}'_{\mathcal{C}} = \{z \in \mathbf{C} \setminus \{0\} : \exists K \in \mathcal{C} : \Delta_K(z) = 0, K \text{ knot and } \Delta_K \text{ square-free}\}.$$

Then $\mathcal{Z}_{\mathcal{C}} \subset \overline{\mathcal{Z}'_{\mathcal{C}}}$ (where bar denotes closure) for $\mathcal{C}$ being one of the classes of non-split (a) *alternating links,* (b) *positive links, or* (c) *special alternating links.*

The paper concludes with some applications of Theorem 1 in §6.

One can (using more advanced perturbation theory for linear operators) give a more direct proof of Theorem 1, which is entirely analytic and unrelated to Proposition 1. The approach taken here replaces most of the analysis by algebra, and is a continuation of our previous study of Tristram-Levine signatures [36]. Since these signatures are predominantly known as concordance invariants, we will build on the work here to address elsewhere concordance issues of special alternating (and more general) knots.

2. PRELIMINARIES ON POLYNOMIALS AND LINK DIAGRAMS

Let $[X]_{t^a} = [X]_a$ be the coefficient of t^a in a polynomial $X \in \mathbf{Z}[t^{\pm 1}]$. For $X \neq 0$, let $\mathcal{C}_X = \{a \in \mathbf{Z} : [X]_a \neq 0\}$ and

$$\min\deg X = \min \mathcal{C}_X, \quad \max\deg X = \max \mathcal{C}_X,$$

$$\operatorname{span} X = \max\deg X - \min\deg X$$

be the minimal and maximal degree and span (or breadth) of X, respectively.

It makes sense to set $\min\deg 0 := \infty$ and $\max\deg 0 := -\infty$. Similarly one defines for $X \in \mathbf{Z}[x_1, \ldots, x_n]$ the coefficient $[X]_Y$ for some monomial Y in the x_i, and $\min\deg_{x_i} X$ etc.

In the sequel the symbol $\subset$ denotes a not necessarily proper inclusion. For a set S, the expression $|S|$ denotes the cardinality of S. Finally, let $\Re e$ and $\Im m$ denote the real and imaginary part, respectively. We will also write $i = \sqrt{-1}$ for the imaginary unit, in situations where no confusion (with its use as an index) arises.

In the sequel $g(K)$ denotes the genus and $\chi(K)$ the Euler characteristic of K (which are the minimal genus resp. maximal Euler characteristic of an orientable spanning surface for K).

A link diagram D is called *split*, or disconnected, if it can be non-trivially separated by a curve in the plane. Else we say the diagram is *non-split*, or connected. A split link is a link with a split diagram. Other links are said to be non-split. We will assume that all links we consider are non-split.

A *region* of a link diagram D is a connected component of the complement of the plane curve of D. A region R of a diagram is called *a Seifert circle region* (resp. non-Seifert circle, or *hole* region), if any two neighboring edges in its boundary (i.e., such sharing a crossing) are equally (resp. oppositely) oriented (clockwise or counterclockwise) as seen from inside R. A diagram is called *special* if and only if all its regions are (either) Seifert circle regions or hole regions.

It is an easy combinatorial observation that for a connected diagram any two of the properties alternating, positive/negative and special imply the third. A diagram with these three properties is called *special alternating*. See e.g. [20]. A special alternating link is a link having a special alternating diagram. It can also be specified (as in the introduction) as a link which is simultaneously positive and alternating. By definition such a link has a positive diagram, and an alternating diagram. That it has a diagram which enjoys both properties simultaneously was proved in [24, 35].

Next we recall that for an alternating link L of $n(L)$ components, the genus $g(L)$ of L coincides with the canonical genus $g(D)$ of an alternating diagram D of L, given by

$$g(D) = \frac{c(D) - s(D) + 2 - n(D)}{2},$$

with $c(D)$, $s(D)$ and $n(D) = n(L)$ being the number of crossings, Seifert circles and components of D, respectively.

3. Alexander-Conway polynomial and Tristram-Levine signatures

We shall briefly introduce the main notions appearing in the sequel. (We also give a few additional references for further details; in particular the reader may consult [36].)

Recall that if M is a Seifert matrix of size $2g \times 2g$ corresponding to a genus g Seifert surface of a knot K, then for any $\xi \in \mathbf{C}$ with $|\xi| = 1$ and $\xi \neq 1$ we define

$$M_\xi(K) := (1-\xi)M + (1-\bar{\xi})M^T,$$

where the bar denotes conjugation and $()^T$ transposition. This is a Hermitian matrix, and all its eigenvalues are real. By $\sigma(M_\xi)$ and $n(M_\xi)$ we denote the signature (sum of signs of eigenvalues) and nullity (number of zero eigenvalues) of M_ξ. They turn out to be independent of the surface and Seifert matrix, and are thus invariants of K, denoted by $\sigma_\xi(K)$ and $n_\xi(K)$ respectively. $\sigma_\xi(K)$ is called a generalized or *Tristram-Levine signature* [37, 14]. It satisfies, as does the usual (Murasugi) signature $\sigma = \sigma_{-1}$ [20], the rules

$$\begin{aligned} \sigma_\xi(L_+) - \sigma_\xi(L_-) &\in \{0,1,2\}, \\ \sigma_\xi(L_\pm) - \sigma_\xi(L_0) &\in \{-1,0,1\}, \\ \sigma_\xi(!L) &= -\sigma_\xi(L), \\ \sigma_\xi(L\#K) &= \sigma_\xi(L) + \sigma_\xi(K). \end{aligned} \tag{1}$$

(Whether to have $\{0,1,2\}$ or $\{0,-1,-2\}$ in (1) is a matter of convention.) Here L_+, L_0, L_- form a skein triple

L_+ L_- L_0

and $!L$ is the mirror image of L. By $K_1 \# K_2$ we denote the connected sum of K_1 and K_2, and $\#^n K$ stands for the connected sum of n copies of K.

The main difference between the general σ_ξ and the usual signature σ is that σ_ξ may take odd values also on knots, and that nice combinatorial formulas, as in the case of alternating links (see [20, 13, 7]), are lacking. Via the Tristram–Murasugi inequality [37, 20], these signatures are related to the 4-genus, and hence to the unknotting number. More recently they have been of some interest because of their relation to the classification of zero sets of algebraic functions on projective spaces [25] and (a quantum version of) the Jones polynomial [6].

The (normalized) *Alexander polynomial* [1] can be defined from a Seifert matrix M by

$$\Delta_K(t) = t^{-g} \det(M - tM^T).$$

Δ satisfies the skein relation

(2) $\Delta\Big($  $\Big) - \Delta\Big($ $\Big) = (t^{1/2} - t^{-1/2})\,\Delta\Big($  $\Big)$,

which defines it alternatively (up to a factor, fixed by requiring that $\Delta(\bigcirc) = 1$).

We will sometimes modify Δ up to units in $\mathbf{Z}[t, t^{-1}]$, as in the original definition of Alexander.

Let ∇_K denote the *Conway polynomial* [2], given by

$$\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t). \tag{3}$$

Consequently, ∇ also satisfies a skein relation, namely

(4) $\nabla\Big($  $\Big) - \nabla\Big($ $\Big) = z\,\nabla\Big($  $\Big)$.

Recall also that for any alternating link L, and in the normalization of Δ_L given by $\min\deg\Delta_L = 0$, we have

$$1 - \chi(L) = \max\deg\Delta_L = \max\deg\nabla_L. \tag{5}$$

See [4, 21]. For a link of $n(L)$ components,

$$2 - n(L) - \chi(L) = 2g(L).$$

Since $\min\deg\nabla_L = n(L) - 1$, we have

$$\operatorname{span}\nabla_L = \max\deg\nabla_L - \min\deg\nabla_L = 2g(L). \tag{6}$$

Theorem 1 originates from a close relationship between the signature and the number of zeros of the Alexander polynomial on the unit circle. This relationship was (possibly first) observed by Riley in the knot case, where it was claimed (see [19, Proposition 1]) that

$$\left|\{\text{zeros of } \Delta_K \text{ on } S^1 \cap \{\Im m\, z > 0\}\}\right| \geq \tfrac{1}{2}\left|\sigma(K)\right|, \tag{7}$$

where $S^1 := \{z : |z| = 1\} \subset \mathbf{C}$ and the zeros are counted with multiplicity. Inequality (7) requires the smoothness (in ξ) of branches of the eigenvalues α_i of the generalized Seifert forms M_ξ. That the α_i can be chosen smoothly was, however, not carefully argued about in any written account I know related to the subject. (The problem occurs at branch intersections where the implicit function theorem fails to ensure smoothness.) This originally motivated the work on

Theorem 1. Only quite a while after this work was done, I understood that the smoothness problem of the α_i can be analytically remedied (using perturbation theory for linear operators, see Theorem 1.10 in Chapter II of [12]). Even so, the proof here is different, and entails steps of separate interest. In particular Proposition 1 has some independent meaning (also for example with regard to Hoste's aforementioned conjecture), and seems not evident analytically. Here analysis will be replaced to large extent by algebra, and the analytic tools required are restricted to Cauchy's integral formula and the implicit function theorem.

The aim of the next sections is to prove inequality (7) in the case of special alternating links, and thus Theorem 1. We will prepare some tools to solve this problem.

4. Some lemmas

We start with several, at first sight apparently unrelated lemmas. The first one comes from number theory and is related to the irreducibility theorem of Hilbert (see e.g. the fourth chapter of [28]): If a polynomial $F(x,y) \in \mathbf{Z}[x,y]$ is irreducible over $\mathbf{Q}$ as a polynomial in two variables, then $F(x,n) \in \mathbf{Z}[x]$ is irreducible over $\mathbf{Q}$ for infinitely many integers n. (Note that for a one variable polynomial in $\mathbf{Z}[x]$ irreducibility over $\mathbf{Q}$ and $\mathbf{Z}$ are equivalent by a classical theorem of Gauss, see e.g. Proposition 2.4 of [31].)

Lemma 1. *If $A, B \in \mathbf{Z}[x]$ are polynomials, such that $A + nB$ is not square-free for any integer $n > 0$, then it is also not square-free for $n \leq 0$.*

Proof. First assume P and Q are coprime polynomials in $\mathbf{Z}[x]$. We want to show that $P + nQ$ is irreducible for infinitely many *positive* integers n. (That it is irreducible for infinitely many integers n follows directly from Hilbert's theorem.)

For this consider $F(x,n) = P(x)+n^2Q(x)$. We are through if it is irreducible. If $F = F_1 \cdot F_2$ (in a non-trivial way), the coprimality of P and Q implies that F_1 and F_2 are both linear in n. Then the vanishing of the n-linear term in F implies that P and Q are up to sign coprime squares. By this argument we would be through when $P + kQ$ is not a square up to sign for some $k > 0$ (replace n^2 by $n^2 + k$ in the definition of F).

Thus assume $P + kQ = \pm L_k^2$ for any $k > 0$. Since $Q \neq 0$, the sign stabilizes for k large enough (it is then positive, say), and beyond that point all L_k are distinct. But then

$$Q = L_{k+1}^2 - L_k^2 = (L_{k+1} + L_k)(L_{k+1} - L_k),$$

so that the L_k can be recovered from factorizations of Q, of which there are only finitely many, a contradiction. Therefore, $P + nQ$ is irreducible for infinitely many $n > 0$.

Now let $S = \gcd(A, B)$. If S is not square-free, then we are done. Otherwise, apply the above argument to $P = A/S$ and $Q = B/S$. Then infinitely many of the $(A + nB)/S$ are irreducible, and discarding the finitely many cases, where they are (irreducible) divisors of S, we find that $A + nB$ is square-free in contradiction to our assumption. This proves the lemma. □

The next lemma is analytic. The notation $f_n \to f$, for functions f_n, f defined on some domain $\mathcal{K}$ and with $n \to \infty$, indicates that $f_n(x) \to f(x)$ for all $x \in \mathcal{K}$ (pointwise convergence), while $f_n \rightrightarrows f$ indicates that $\forall \varepsilon > 0\ \exists n_\varepsilon\ \forall n > n_\varepsilon\ \forall x \in \mathcal{K} : |f_n(x) - f(x)| < \varepsilon$ (uniform convergence).

LEMMA 2. *Assume f_n and f are analytic functions on some open set $\mathcal{O} \subset \mathbf{C}$ and $f \not\equiv 0$. Let $f_n \rightrightarrows f$ converge uniformly on each compact subset $\mathcal{K} \subset \mathcal{O}$. Let z be a zero of f of finite order. Then there are zeros z_n of f_n with $z_n \to z$.*

Proof. By analyticity z is an isolated zero of f, so

$$\exists \varepsilon > 0 : f|_{B(z,\varepsilon)\setminus\{z\}} \neq 0.$$

(Here $B(z, \varepsilon) = \{ z' \in \mathbf{C} : |z - z'| < \varepsilon \}$ and $f \neq 0$ means that f has no zero.) Fix some $\varepsilon' \in (0, \varepsilon]$. Since $f_n \rightrightarrows f$, for n large enough $f_n|_{\partial B(z,\varepsilon')\setminus\{z\}} \neq 0$. Let s be the order of z. Then, choosing the same branch of the s-th root, $f_n^{-1/s} \rightrightarrows f^{-1/s}$ on $\partial B(z, \varepsilon')$, and hence

$$\oint_{\partial B(z,\varepsilon')} f_n^{-1/s}\, dz \longrightarrow \oint_{\partial B(z,\varepsilon')} f^{-1/s}\, dz.$$

If now there is a sequence $\{n_i\}$ such that f_{n_i} has no zero in $B(z, \varepsilon')$, then

$\oint_{\partial B(z,\varepsilon')} f_{n_i}^{-1/s}\, dz \equiv 0$, since $f_{n_i}^{-1/s}$ is holomorphic on $B(z,\varepsilon')$. But

$$\oint_{\partial B(z,\varepsilon')} f^{-1/s}\, dz = 2\pi i \operatorname{res}_{f^{-1/s}}(z) \neq 0\,,$$

a contradiction. $\square$

The third preparatory lemma is a special case of what we want to prove. Denote by $f(\xi) \sim g(\xi)$ the property $\lim f(\xi)/g(\xi) = 1$, when ξ converges to a limit specified from the context. The symbol $o(\xi)$ denotes a function with $0 \sim o(\xi)$. The term "double zero" always (also later) means at least double, i.e., a multiple zero.

LEMMA 3. *Let K be a knot, such that ∇_K has no double zero. Then inequality (7) holds.*

Proof. The statement is equivalent to saying that if Δ_K has no double root, at least $|\sigma/2|$ of its roots lie on the upper (positive imaginary part) half-arc of S^1.

Take a Seifert matrix M of K of size $n = 1 - \chi(K) = 2g(K)$. Then consider

$$M_\xi = (1-\xi)M + (1-\bar{\xi})M^T$$

for $\xi \in S^1 \cap \{\, \Im m\ z > 0 \,\}$. Since

$$\Delta_K(t) = \det(M - tM^T)\,,$$

and

$$\frac{\bar{\xi}-1}{1-\xi} = \frac{\xi^{-1}-1}{1-\xi} = \frac{1}{\xi}\,,$$

we have

$$\det(M_\xi) = (1-\xi)^n\, \Delta_K\left(\frac{\bar{\xi}-1}{1-\xi}\right) = (1-\xi)^n\ \ \Delta_K(\xi^{-1}).$$

Let $\chi_p(M_\xi)$ be the characteristic polynomial (considered as a polynomial in one complex variable α) of M_ξ. Then $\det(M_\xi) = \chi_p(M_\xi)(0)$. We argued that the function $f(\xi) = \det(M_\xi) = \chi_p(M_\xi)(0)$ has a non-zero derivative in each zero ξ_0. Then by the implicit function theorem, there is an $\varepsilon > 0$ and a function

$$g\colon\ (-\varepsilon,\, \varepsilon) \longrightarrow (\xi_0 e^{-2\pi i\varepsilon},\, \xi_0 e^{2\pi i\varepsilon})$$

(the interval on the right being meant as an arc on S^1), such that $g(0) = \xi_0$ and $\chi_p(M_\xi)(\alpha) = 0 \Leftrightarrow \xi = g(\alpha)$. This shows that for ξ around ξ_0, there is a unique (still possibly multiple) eigenvalue α of M_ξ around 0. Since f changes sign, so must α, and it must be of odd multiplicity.

Assume now that this multiplicity is $k > 1$. Set $\alpha = \alpha_1 = \cdots = \alpha_k$. Since

$$f(\xi) \underset{\xi \to \xi_0}{\sim} D \cdot (\xi - \xi_0),$$

with $D = \dfrac{\partial f}{\partial \xi}(\xi_0) \neq 0$, we must have

$$\alpha_i \sim D' \cdot \sqrt[k]{\xi - \xi_0}, \quad i \leq k$$

with $D' = \sqrt[k]{D} \neq 0$. We also have that α_i for $i > k$ are continuous in ξ and bounded away from 0 as $\xi \to \xi_0$. Then consider

$$\chi_p(M_\xi)(x) = \prod_{i=1+k}^{n} (x - \alpha_i) \cdot \left(x - D' \sqrt[k]{\xi - \xi_0} + o\left(\sqrt[k]{\xi - \xi_0}\right)\right)^k, \tag{8}$$

with $o(\ldots)$ meant w.r.t. the asymptotical behavior as $\xi \to \xi_0$. The first factor

$$\hat{\chi}(x) = \prod_{i>k} (x - \alpha_i)$$

must have absolute coefficient bounded away from 0 for $\xi \to \xi_0$. By the same argument we have that $[\chi_p]_k$ is bounded away from 0.

Now we find for $m \leq n - k$

$$[\hat{\chi}]_m = [\chi_p]_{m+k} + [\chi_p]_{m+k+1} \cdot kD' \sqrt[k]{\xi - \xi_0} + o\left(\sqrt[k]{\xi - \xi_0}\right). \tag{9}$$

Here $[\hat{\chi}]_l$ and $[\chi_p]_l$ are to be interpreted as the coefficients of x^l in $\chi_p(x) = \chi_p(x, \xi)$, $\hat{\chi}(x) = \hat{\chi}(x, \xi) \in \mathbf{C}[x]$, regarded as functions of ξ.

A way to see (9) directly is to rewrite (8) as

$$\chi_p(x) = \hat{\chi}(x) \cdot x^k \left(1 - kD' \sqrt[k]{\xi - \xi_0} \cdot \frac{1}{x} + o[x]\left(\sqrt[k]{\xi - \xi_0}\right)\right),$$

where $o[x](\sqrt[k]{\xi - \xi_0})$ stands for a polynomial in $1/x$ all whose coefficients are $o(\sqrt[k]{\xi - \xi_0})$. Then

$$\hat{\chi}(x) = \chi_p(x) \cdot x^{-k} \left(1 + kD' \sqrt[k]{\xi - \xi_0} \cdot \frac{1}{x} + o[[x]]\left(\sqrt[k]{\xi - \xi_0}\right)\right), \tag{10}$$

where $o[[x]]\left(\sqrt[k]{\xi - \xi_0}\right)$ stands for a power series in $1/x$ which converges absolutely for $|x| \geq 1$ when ξ is close to ξ_0, and whose coefficients are $o\left(\sqrt[k]{\xi - \xi_0}\right)$ when $\xi \to \xi_0$. Taking the coefficients in (10) we obtain (9). (Another way to see (9) is to compare iteratedly in (8) coefficients of x^n, x^{n-1}, x^{n-2} etc.)

But for $m = -1$ we have $[\hat{\chi}]_m \equiv 0$, while $[\chi_p]_{m+k}$ and $[\chi_p]_{m+k+1}$ are smooth and the latter is bounded away from 0, so that (9) cannot hold for $\xi \to \xi_0$. (The term $\sqrt[k]{\xi - \xi_0}$ is dominating in the ξ-derivative, contradicting the smoothness of the left-hand side in ξ.) Thus $k = 1$.

Therefore, as desired, at each zero of Δ on the upper half-arc of S^1, exactly one eigenvalue of M_ξ (now counted *with* multiplicity) changes sign. Then σ_ξ changes by ± 2 each time. Now $\sigma_{-1} = \sigma$ and $\sigma_1 = 0$ because $M_1 = 0$. Unfortunately, this form is singular, so *a priori* σ_ξ may change around $\xi = 1$. If we ensure that it does not, then we obtain that an eigenvalue of M_ξ must change sign on an arc between $\xi = -1$ and $\xi = 1$ at least $|\sigma/2|$ times, and the result follows.

So we will conclude by arguing that σ_ξ does not change around $\xi = 1$ (i.e. remains 0 for ξ close to 1). This becomes clear once we see that the singularity $\xi = 1$ of M_ξ can be avoided. To do so, consider $M'_\xi = M_\xi/|1-\xi|$ for $\xi \neq 1$. Now we have

$$\frac{1-\xi}{|1-\xi|} = \sqrt{\frac{1-\xi}{1-\bar{\xi}}} = \sqrt{\frac{1-\xi}{1-\xi^{-1}}} = \sqrt{-\xi} = \pm i\,\xi^{1/2}, \tag{11}$$

and looking at $\xi \to 1$ (while fixing the branch of the square root so that also $\xi^{1/2} \to 1$) reveals that the correct sign is the positive one. By inverting (11), or by a similar calculation and limit argument, we see that

$$\frac{1-\bar{\xi}}{|1-\xi|} = -i\,\xi^{-1/2}.$$

So $M'_\xi = i\,\xi^{1/2}M - i\,\xi^{-1/2}M^T$, which has a smooth continuation in $\xi = 1$, and M'_1 has determinant

$$\det(M'_1) = i^n \det(M - M^T) = i^n\,\Delta_K(1) = i^n = (-1)^{g(K)},$$

so is regular. Moreover, $M_1'^{\,T} = -M'_1$, so that $\sigma(M'_1) = 0$, and clearly $\sigma(M_\xi) = \sigma(M'_\xi)$ for $\xi \neq 1$. □

COROLLARY 1. *Let K be a special alternating knot, such that ∇_K has no double zero. Then all zeros of $\nabla_K(\sqrt{z})$ lie in $[-4, 0]$.*

Proof. We have $\sigma(K) = 2g(K)$ by [20], and equality (5) holds. □

5. Proof of Theorem 1

We will now establish the general case of Theorem 1. It follows directly from Corollary 1 and Proposition 1. So we prove Proposition 1. For the proof the notion of braiding sequences [33] is used.

Let $\widehat{D}$ be an oriented link diagram and P a set of crossings in $\widehat{D}$, which we call marked and number $c_1, \ldots, c_n$. One can consider the family of diagrams

$$\mathcal{B} = \{\widehat{D}(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbf{Z} \text{ odd}\}.$$

Here the diagram $\widehat{D}(x_1, \ldots, x_n)$ is obtained from $\widehat{D}$ by replacing the crossing c_i by a tangle of $|x_i|$ reverse half-twists of sign $\operatorname{sgn}(x_i)$:

$x_i = -3$ $\quad$ $x_i = -1$ $\quad$ $x_i = 1$ $\quad$ $x_i = 3$

In [33] we called $\mathcal{B}$ the *braiding sequence* $\mathcal{B}(\widehat{D}, P)$ associated to $(\widehat{D}, P)$ with antiparallel braidings at each crossing in P, parametrized by $n = |P|$ odd integers $x_1, \ldots, x_n$. We shall use this terminology from now on.

Lemma 4. *Let D be a positive and/or alternating link diagram. Then there exists a braiding sequence $\mathcal{B} = \mathcal{B}(\widehat{D}, P)$ with the following features:*

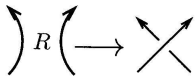
- *$\widehat{D}$ is a positive and/or alternating knot diagram,*
- *$\mathcal{B}$ contains an unknot diagram, and*
- *when smoothing out in $\widehat{D}$ all marked crossings $c_i \in P$, and removing nugatory crossings using Reidemeister I moves, we obtain D.*

Proof. If D is not already a knot diagram, we describe first a way to make D into a knot diagram D', preserving the property to be positive and/or alternating (otherwise set $D' = D$).

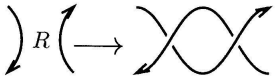
If D is special alternating, it has 2 types of regions: Seifert circle regions and non-Seifert circle regions. Since each edge (piece of a strand between two crossings) is bounded by a non-Seifert circle region, it is easy to see that D has a non-Seifert circle region R bounded by edges $e_{1,2}$ belonging to different components. If D is not special alternating, choose R to possess proper $e_{1,2}$ but without the requirement to be a non-Seifert circle region.

The orientation of $e_{1,2}$ may be parallel or reverse as seen from inside R.

1. If the orientation is parallel, apply the move

(12) 

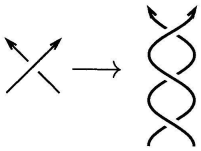
2. If the orientation is reverse, apply the move

(13) 

Since the move (12) does not create a new Seifert circle, and the move (13) creates a Seifert circle with empty interior, both moves preserve specialty of the diagram, and if the crossings are appropriately chosen, also positivity and/or alternation. Furthermore, the moves decrease the number of components of the link by one. (Thus they augment the canonical genus of the diagram by one.)

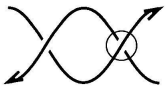
Repeating this procedure, we arrive at a positive and/or alternating knot diagram D'.

Now D' may be unknotted by switching some crossings $d_1, \ldots, d_k$. At each such crossing, make a replacement

(14) 

One obtains an unknot diagram D''. The move (14) also augments the canonical genus of the diagram by one, and at least one such move is needed (because positive or alternating diagrams are not unknotted), so we have $g(D'') > 0$. (Alternatively, using (6), we see that $g(D'') = 0$ implies that ∇_L is a single monomial, and there would be nothing to prove in Theorem 1.)

Now mark in D'' a set P of crossings as follows:

- the crossings created by (12): 
- *one* of the crossings created by (13): 
- the two (lower) crossings created by (14): 

Let $\widehat{D}$ be obtained from D'' by switching crossings in P so that $\widehat{D}$ becomes positive and/or alternating. Then $\mathcal{B} = \mathcal{B}(\widehat{D}, P)$ has all the stated properties. $\square$

Proof of Proposition 1. Assume (after possibly taking the mirror image, which does not change the zeros of ∇) that L is a positive and/or alternating link and $z_0 \in \mathbf{C}\backslash\{0\}$ with $\nabla_L(\sqrt{z_0}) = 0$. Let D be a positive and/or alternating diagram of L. Then we consider the braiding sequence $\mathcal{B} = \mathcal{B}(\widehat{D}, P)$ found in Lemma 4. The desired conclusion will be obtained by proving algebraically that enough of the ∇ polynomials on $\mathcal{B}$ are square-free and approximating z_0 by zeros of such polynomials.

Now consider the map

$$(15) \qquad (2\mathbf{Z}+1)^n \ni (x_1, \dots, x_n) \longmapsto \nabla\big(\widehat{D}(x_1, \dots, x_n)\big)(z) \in \mathbf{Z}[z]\,.$$

Several properties of this map can be seen by applying iteratedly the skein relation (4) for ∇. An explicit account may be found in [29]. (See in particular Theorem 3.4 (2) and its proof. Note, though, that we do not consider parallel twists, which are also treated there, and the formulation for the Alexander polynomial is given, equivalent up to the change of variables (3).) We know that (15) becomes a polynomial X in $x_1, \dots, x_n$, with coefficients in $\mathbf{Z}[z]$. (For the explanation we will just make it is useful to neglect the polynomial structure of X in z and regard z as just a quantity entering into the coefficients w.r.t. the x_i.) Furthermore, X is at most linear in each x_i (and so of total degree at most n). We call a monomial $Q = x_{i_1} \dots x_{i_k}$ (with i_j pairwise distinct) in X maximal, if Q does not divide (properly) any other monomial in X. (Note that $k \le n$, and if $k = n$ then Q is always maximal.) If Q is maximal, and $\widetilde{D}$ is the diagram obtained by smoothing out the k crossings c_{i_j} in $\widehat{D}$, then the coefficient of Q in X is given by $[X]_Q = (z/2)^k \nabla(\widetilde{D})$.

Assume that $\widehat{D}(x_1, \dots, x_n)$ is positive and/or alternating for $x_1, \dots, x_n > 0$. (By convention, this is always so, unless $\widehat{D}$ is alternating, but not positive. In latter case, the assumption is to be understood so that we change the sign of some x_i in the argument to follow.) For $x_1, \dots, x_n > 0$ consider $\nabla\big(\widehat{D}(x_1 p, \dots, x_n p)\big)$ with $p > 0$ odd. By the above remark, this is a polynomial in p. Its leading p-coefficient is $C(z)\nabla_L(z)$, with $C(z) = (z/2)^n x_1 \cdot \dots \cdot x_n$. The other factor comes from the Conway polynomial of the link obtained by smoothing out all c_i, which (by Lemma 4) is L. Thus as $p \to \infty$ odd,

$$\frac{1}{p^n} \nabla\big(\widehat{D}(x_1 p, \dots, x_n p)\big)(\sqrt{z}) \sim C(\sqrt{z})\nabla_L(\sqrt{z}) + o(1)\,,$$

where $o(1)$ stands for a sequence of polynomials of bounded (z-)degree with coefficients going to zero. This implies that

$$\frac{1}{p^n}\nabla\big(\widehat{D}(x_1p,\dots,x_np)\big)(\sqrt{z}) \rightrightarrows C(\sqrt{z})\nabla_L(\sqrt{z})$$

on any compact set $\mathcal{K}$ (when $p\to\infty$). Since $C(\sqrt{z})\neq 0$ for z close to z_0, by Lemma 2

$$P_p(z)=\nabla\big(\widehat{D}(x_1p,\dots,x_np)\big)(\sqrt{z})$$

has zeros converging to z_0. Thus we would be done, unless for any choice of $x_1,\dots,x_n>0$, the polynomial $P_p(z)$ has a double z-zero for all sufficiently large p. Now, assume it were so.

Fix $x_1,\dots,x_n>0$. The existence of a double zero in P_p is equivalent to the vanishing of the discriminant $\operatorname{disc}(P_p)$ of P_p. The discriminant $\operatorname{disc}(f)=\operatorname{Res}(f,f')$, where $\operatorname{Res}(f,g)$ for $f,g\in\mathbf{Z}[z]$ is the resultant. This resultant can be expressed as the determinant of a square matrix $M_{f,g}$ of dimension $\deg_z f+\deg_z g$, involving the coefficients of f and g, see e.g. [15, definition 1.93, p. 36]. Since $\max\deg_z P_p=g(\widehat{D})$ for $p>0$, the size of $M_{P_p,\partial P_p/\partial z}$ is $2g(\widehat{D})-1$, constantly in p. (To avoid degeneracies we excluded $g(\widehat{D})=0$ in the proof of Lemma 4.) We know also that the function

$$p\longmapsto\big[\nabla\big(\widehat{D}(x_1p,\dots,x_np)\big)\big]_i,$$

$[\,.\,]_i$ denoting the coefficient in z^i, is a polynomial in p for any i. Thus the entries of $M_{P_p,\partial P_p/\partial z}$ are polynomial in p. So $\operatorname{disc}(P_p)$ is a polynomial in p for $p>0$. But, as we saw, this polynomial vanishes for $p\to\infty$, and hence it vanishes identically, in particular for $p=1$.

This means that $\nabla\big(\widehat{D}(x_1,\dots,x_n)\big)$ has a double zero for any $x_1,\dots,x_n>0$, and hence is divisible by the square of its minimal polynomial. We claim that then $\nabla\big(\widehat{D}(x_1,\dots,x_n)\big)$ has a double zero for any odd (not necessarily positive) $x_1,\dots,x_n$. To see this, use induction on the number of negative parameters x_i, the linear dependence of $\nabla\big(\widehat{D}(x_1,\dots,x_n)\big)$ on each x_i (when fixing the others), and Lemma 1. But $\mathcal{B}$ contains an unknot diagram (by Lemma 4), whose polynomial $\nabla=1$ is certainly square-free.

This (indirect) argument provides us with the desired contradiction, originating from our assumption that almost all P_p are not square-free, and thus completes the proof. $\square$

REMARK 1. One sees from the proof of Lemma 4 and Proposition 1 that they work also for arborescent links, but in that case the proposition is not helpful, since H. Murakami gave a proof that all (admissible) Alexander polynomials are realized by arborescent knots [17].

6. Some inequalities

With Theorem 1 proved, it is worth remarking that it implies several further inequalities. For example, we have

Corollary 2. *For a special alternating link L and $i \leq 1 - \chi(L)$,*

$$[\nabla_L]_{i-2} \leq -\chi(L)[\nabla_L]_i . \tag{16}$$

Proof. Let $n(L)$ be the number of components of L. Using Theorem 1, Newton's theorem, and the positivity of all $[\nabla_L]_i$ (see [3, 34]), we have for $1 + n(L) \leq i \leq -1 - \chi(L)$ and $i + \chi(L)$ odd

$$\frac{[\nabla_L]_{i-2}}{[\nabla_L]_i} < \frac{[\nabla_L]_i}{[\nabla_L]_{i+2}} . \tag{17}$$

Thus it suffices to prove (16) for $i = 1 - \chi(L)$. In this case (16) is equivalent to the inequality $[\Delta_L]_0 \leq -[\Delta_L]_1$, with Δ_L normalized so that $\min \deg \Delta_L = 0$ and $[\Delta_L]_0 > 0$, since (in this normalization of Δ_L) we have

$$\Delta_L(t) = \nabla_L\left(t^{-1/2} - t^{1/2}\right) \cdot t^{(1-\chi(L))/2} . \tag{18}$$

The desired inequality was proved in [23, Proposition 2]. □

Remark 2. The inequality in the corollary is sharp (non-trivially, that is, for $1 - \chi(L) \geq i \geq n(L) + 1$ and $i + \chi(L)$ odd) if and only if $i = 1 - \chi(L)$ and L is a $(2, n)$-torus knot or link (with both orientations in the link case). This follows from the remark after the proof of Proposition 2 of [23]. (Note that not all graphs given there are relevant for link diagrams.)

Another inequality, using the specific range of zeros of $\nabla(\sqrt{z})$, is a lower estimate for the coefficients of ∇ in terms of the genus.

Corollary 3. *For a special alternating link L we have*

$$[\nabla_L]_{n(L)-1+2i} \geq \frac{1}{4^i} \binom{g(L)}{i} [\nabla_L]_{n(L)-1} . \tag{19}$$

Note that the only previous results in this direction (albeit true more generally for positive links) were that these coefficients are non-negative [3], with the slight improvement in [34] that (for $i \leq g(L)$) they are strictly positive, and the inequalities for knots ($n(L) = 1$) and $i = 1$ of [32].

Proof. Use that $[\nabla_L]_{n(L)-1} \geq 1$ and that for $z_1, \ldots, z_{g(L)}$ being the zeros of $\nabla_L(\sqrt{z})$, we have

$$\frac{[\nabla_L]_{n(L)-1+2i}}{[\nabla_L]_{n(L)-1}} = (-1)^i \sum_{\substack{I \subset \{1,\ldots,g(L)\} \\ |I|=i}} \prod_{i \in I} \frac{1}{z_i}. \qquad \square$$

We conclude by giving a few examples illustrating these inequalities. One can find alternating knots which can be excluded from being special alternating using Corollary 2.

EXAMPLE 1. The alternating knots 14_{10475} and 14_{12515} of [11] have the Conway polynomials $1 + 8z^2 + z^4 + z^6 + 2z^8$ and $1 + 3z^2 + 8z^4 + z^6 + 2z^8$, respectively, and $\chi = -7$. The polynomials are positive, so that by [3] the knots may be positive, and so (as explained, see [24, 35]) special alternating. But they are not special alternating by Corollary 2.

For Corollary 3 we have the following

EXAMPLE 2. The knots 12_{1221} and 12_{1238} of [11] are alternating. Their Conway polynomials are $1 + z^2 + 4z^4 + 8z^6 + 5z^8 + z^{10}$ and $1 + z^2 + 8z^4 + 12z^6 + 6z^8 + z^{10}$, and genus $g = 5$. Their polynomials are positive, but violate (19) for $i = 1$. The alternating knot 16_{377355} has $g = 7$ and $\nabla(z) = 1 + 3z^2 + z^4 + 12z^6 + 28z^8 + 23z^{10} + 8z^{12} + z^{14}$, which violates (19) for $i = 2$.

REMARK 3. Conjecture 1 implies that Corollary 2 may hold also for positive links. But Corollary 3 does make use of the range of the zeros of ∇ (rather than only the fact that they are real), and may not hold for some positive links even assuming Conjecture 1. Nevertheless, we do not know of any such links.

REMARK 4. Newton's theorem shows in fact a stronger version of the property 'Newton-like', involving combinatorial coefficients in the inequalities (see Theorem 51 of [8]), but we preferred to omit these coefficients in order to have a stronger analogy to the property 'weakly Newton-like' (where they cannot appear).

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