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COUNTING SOLUTIONS OF PERTURBED HARMONIC MAP EQUATIONS

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ABSTRACT. In this paper we consider perturbed harmonic map equations for maps between closed Riemannian manifolds. In the case where the target manifold has negative sectional curvature we prove - among other results - that for a large class of semilinear and quasilinear perturbations, the perturbed harmonic map equations have solutions in any homotopy class of maps for which the Euler characteristic of the set of harmonic maps does not vanish. Under an additional condition, similar results hold in the case where the target manifold has nonpositive sectional curvature. The proofs are presented in an abstract setup suitable for generalizations to other situations.

1. INTRODUCTION

In this work we study the solvability of a class of semilinear and quasilinear perturbations of the harmonic map equation for maps $u: M \longrightarrow M'$ from a closed n-dimensional Riemannian manifold M into a closed n'-dimensional Riemannian manifold M' with nonpositive sectional curvature. Denoting points of M and M' by x and y, respectively, we study the set of solutions for the equations

$$\tau(u)(x) + F(x, u(x)) = 0 \tag{1.1}$$

and

$$\tau(u)(x) + F(x, u(x)) + d_x u(G(x, u(x))) = 0 \tag{1.2}$$

where $\tau(u)(x)$ denotes the tension field, F is an x-dependent vector field on M' and G is a y-dependent vector field on M. Recall that the harmonic map equation $\tau(u) = 0$ is the Euler-Lagrange equation of the energy functional

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$$E(u) = \frac{1}{2}\int_M \|d_x u\|^2 \,\mathrm{dvol}(x)\,. \tag{1.3}$$

Solutions of $\tau(u) = 0$ are therefore critical points of $E(u)$. They are referred to as harmonic maps. In local coordinates of M and M', the components $\tau^\alpha(u)(x)$ of the tension field $\tau(u)$ are given by ($1 \le \alpha \le n'$)

$$\tau(u)^\alpha(x) = \Delta_M u^\alpha(x) + \sum_{1\le i,j\le n} g^{ij}(x) \sum_{1\le\beta,\gamma\le n'} \Gamma'^\alpha_{\beta,\gamma}(u(x))\frac{\partial u^\beta(x)}{\partial x^i}\frac{\partial u^\gamma(x)}{\partial x^j}$$

where Δ_M is the Laplace-Beltrami operator on M, $g^{ij}(x)$ are the components of the inverse of the metric tensor on M, and $\Gamma'^\alpha_{\beta,\gamma}(y)$ denote the Christoffel symbols corresponding to the Riemannian metric g' on M'.

We point out immediately that the perturbations considered are not necessarily of variational type, meaning that for generic F and G the equations (1.1) and (1.2) are not the Euler-Lagrange equations of any perturbation of the energy functional $E(u)$. Our studies were motivated in part by work of Kuksin [Ku] on perturbed Cauchy-Riemann equations.

To state our results, we need to introduce some notation. For any $j > n/2$ we denote by $\mathcal{H}^{(j)}$ the Hilbert manifold of maps from M to M' of Sobolev class H^j. By the Sobolev embedding theorem, the space $\mathcal{H}^{(j)}$ compactly embeds into $C^i(M, M')$ for every $0 \le i < j - n/2$. The connected components $\mathcal{H}^{(j)}_\zeta$ of the space $\mathcal{H}^{(j)}$ correspond to the homotopy classes ζ of maps from M to M'. We now fix an integer $k > 2 + \frac{n+n'}{2}$ and denote by $\mathcal{F}^{(k)}$ the space of x-dependent vector fields on M' of class H^k, and by $\mathcal{G}^{(k)}_c$ the space of y-dependent vector fields G on M of class H^k satisfying

$$\|G\| := \sup_{x\in M, y\in M'} \|G(x,y)\| < c\,.$$

We consider the set

$$\mathcal{M}^{(k)}_\zeta := \left\{(u,F) \in \mathcal{H}^{(k+2)}_\zeta \times \mathcal{F}^{(k)} \mid (u,F) \text{ solves } (1.1)\right\}$$

of solutions (u, F) of (1.1) and, for any $c > 0$, the set

$$\mathcal{N}^{(k)}_{\zeta,c} := \left\{(u,F,G) \in \mathcal{H}^{(k+2)}_\zeta \times \mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c \mid (u,F,G) \text{ solves } (1.2)\right\}$$

of solutions (u, F, G) for equation (1.2).

Our goal here is to give a sufficient criterion for the solvability of equations (1.1) and (1.2) for every perturbation $F \in \mathcal{F}^{(k)}$ (resp. $(F, G) \in \mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c$ with $c > 0$ small) and to provide a count for the number of solutions for generic such perturbations.

THEOREM 1.1. *Let ζ be a homotopy class of maps from the closed Riemannian manifold M of dimension n to the closed Riemannian manifold M' of nonpositive sectional curvature and dimension n'. Then there exists an integer $D_\zeta \in \mathbf{Z}$ such that for every $k > 2 + \frac{n+n'}{2}$ the following statements hold:*

1. *If $D_\zeta \neq 0$, then equation (1.1) has at least one solution in ζ for every $F \in \mathcal{F}^{(k)}$. Moreover, for any F contained in an open, dense subset $\mathcal{F}^{(k)}_{reg} \subset \mathcal{F}^{(k)}$ equation (1.1) has at least $|D_\zeta|$ solutions in ζ.*
2. *There exists $c > 0$ such that if $D_\zeta \neq 0$, then equation (1.2) has at least one solution in ζ for any $(F, G) \in \mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c$. Moreover, for any pair (F, G) contained in an open, dense subset $(\mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c)_{reg} \subset \mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c$, equation (1.2) has at least $|D_\zeta|$ solutions in ζ.*

Furthermore, if the critical set of the restriction $E|_\zeta$ of the energy functional (1.3) to $\mathcal{H}^{(k+2)}_\zeta$ is nondegenerate - as is for instance the case for any homotopy class if M' has negative sectional curvature -, then

$$D_\zeta = \chi(Crit(E|_\zeta)), \tag{1.4}$$

the Euler characteristic of the set of harmonic maps in the homotopy class ζ.

The nondegeneracy condition appearing in the last part of Theorem 1.1 is of Morse-Bott type and is discussed in detail in §2. We point out that (1.4) is sharp, in the sense that there are examples of homotopy classes ζ and perturbations F for which the number of solutions of equation (1.1) in ζ equals $|\chi(Crit(E|_\zeta))|$.

The proof of Theorem 1.1 naturally splits into several parts. For the first part, one starts by observing that the $\mathcal{M}^{(k)}_\zeta$ (resp. $\mathcal{N}^{(k)}_{\zeta,c}$) are Hilbert manifolds and that the projections $\pi_{k,\zeta} : \mathcal{M}^{(k)}_\zeta \longrightarrow \mathcal{F}^{(k)}$ (resp. $\pi_{k,\zeta,c} : \mathcal{N}^{(k)}_{\zeta,c} \longrightarrow \mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c$) are Fredholm maps of index 0. The subset $\mathcal{F}^{(k)}_{reg} \subset \mathcal{F}^{(k)}$ (resp. $(\mathcal{F}^{(k)} \times \mathcal{G}^{(k)}_c)_{reg}$) in Theorem 1.1 is precisely the set of regular values of the map $\pi_{k,\zeta}$ (resp. $\pi_{k,\zeta,c}$). The integer D_ζ in the statement of the theorem is nothing but the degree of these maps. In §3 we present a detailed derivation of the existence of such a degree under suitable general conditions. We take a common approach using determinant line bundles defined for a family of Fredholm operators of some fixed index. For the convenience of the reader, we have given a self-contained review of the determinant line bundle in Appendix A.

In the second part of the proof our goal is to compute the degree. This is carried out in §4, where we use the nondegeneracy assumption for the critical set to identify the degree D_ζ with the Euler characteristic of the set of critical points of the functional whose perturbation is studied. Here the basic idea is to reduce the problem to an equivalent finite-dimensional one, where the corresponding result is well-known.

Finally, in §5 we verify the assumptions of the general argument in the previous two sections for the perturbed harmonic map equations (1.1) and (1.2) and prove Theorem 1.1. Here the argument rests on the following compactness result established in [KKS1, Ko]. Recall that a continuous map is said to be proper if the inverse image of every compact set is compact.

THEOREM 1.2. *Assume that M' has non-positive sectional curvature, ζ is a homotopy class of maps from M to M' and that $k > 2 + (n + n')/2$. Then the following statements hold:*

1. *The natural projection onto the second factor $\pi_{k,\zeta}\colon \mathcal{M}_\zeta^{(k)} \longrightarrow \mathcal{F}^{(k)}$ is proper.*
2. *There exists $c > 0$ such that the natural projection*
$$\pi_{k,\zeta,c}\colon \mathcal{N}_{\zeta,c}^{(k)} \longrightarrow \mathcal{F}^{(k)} \times \mathcal{G}_c^{(k)}$$
is proper.

Part (1) of Theorem 1.2 was established in [KKS1], along with part (2) in the case when $\dim M \leq 3$. In the recent paper [Ko], the case $\dim M \geq 3$ of part (2) is treated. Note that in [KKS1, Ko], the results of Theorem 1.2 are stated and proved for C^l maps with $l \geq 2$. The versions above can be proven in essentially the same way. In [KKS2] it is shown that the constant $c > 0$ in Theorem 1.2 (2) can be chosen independently of the homotopy class ζ if M' has strictly negative sectional curvature (see also [Ko]). In fact, it is proved that c can be chosen to depend only on the upper bound of the sectional curvature and on the lower bound of the injectivity radius of M'. We point out that there are examples of large perturbations G such that the set of solutions of equation (1.2) is no longer compact. Hence for sufficiently large c, part (2) of Theorem 1.2 no longer holds.

In order to show that formula (1.4) of Theorem 1.1 holds in many situations, we begin our exposition in §2 by explaining the nondegeneracy condition on the energy functional and proving that it is in particular satisfied for the following triples (M', M, ζ):

- M' has non-positive sectional curvature, M is arbitrary and ζ is the trivial homotopy class.
- $M' = \mathbf{R}^{n'}/\Lambda$ is a flat torus and M and ζ are arbitrary.
- M' has negative sectional curvature and M and ζ are arbitrary.

- $(M_1' \times M_2', M, \zeta_1 \times \zeta_2)$ where both triples, (M_1', M, ζ_1) and (M_2', M, ζ_2), satisfy the non-degeneracy condition.

To illustrate our results, we next present some examples.

EXAMPLES.

1. *Taking $M = M' = S^1 \simeq \mathbf{R}/\mathbf{Z}$, any map $u\colon M \longrightarrow M'$ can be lifted to a map $\widetilde{u}\colon \mathbf{R} \longrightarrow \mathbf{R}$ with*

$$\widetilde{u}(x+1) = \widetilde{u}(x) + d\,, \tag{1.5}$$

where the integer $d \in \mathbf{Z}$ determines the homotopy class $\zeta = [u]$. Crit($E|_\zeta$) is always diffeomorphic to S^1 (see the proof of Proposition 2.7), so that the Euler characteristic vanishes. Equation (1.1) *for $\widetilde{u}$ with a constant, non-zero vector field F reduces to*

$$\widetilde{u}_{xx} + c = 0 \quad , \quad c \neq 0$$

which has no solutions at all satisfying (1.5).

2. *Taking $M' = F_g$, a surface of genus $g > 1$ with constant negative curvature, M arbitrary and ζ the trivial homotopy class, we find that Crit($E|_\zeta$) consists precisely of the constant maps and hence its Euler characteristic equals $2-2g$. Thus according to Theorem 1.1, there is a nullhomotopic solution $u\colon M \longrightarrow F_g$ for equation* (1.1) *for* every *perturbation, and for generic perturbations there are at least $2g-2$ of them.*
3. *Taking M' negatively curved, and M such that there exists a homotopy class ζ of maps from M to M' for which the image of $\pi_1(M)$ in $\pi_1(M')$ under the induced map is not trivial or infinite cyclic, we find that Crit($E|_\zeta$) consists of one point [Ha, SY]. It follows that equation* (1.1) *has a solution in the class ζ for* every *perturbation F.*

We conclude this introduction with a few historical comments. Results of the type stated in Theorem 1.1 have their origin in the fundamental work of Smale [Sm], where he proved the infinite dimensional version of Sard's theorem for Fredholm maps and observed that the (unoriented) cobordism class of the preimage of a regular value of a proper Fredholm map is an invariant directly generalizing the ordinary mod 2 degree. These results have been extended, applied and refined in many contexts. For example, Tromba [Tr] defined an integer degree for proper Fredholm vector fields of index zero (with respect to a connection) on a Banach manifold. We choose a somewhat different, though also quite common approach using determinant line bundles which allows us to deal with a larger class of vector fields. Thus §3 is of expository nature and included mostly for the convenience of the reader. In §4 we present a short and self-contained proof of formula (1.4). We remark that it is implied by the main theorem in a recent paper by Cieliebak, Mundet i Riera and Salamon [CMS], where a framework for results of this sort is developed in a much more general (and complicated) situation. Using their work we arrived at an improved version of our original approach.

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2. Nondegeneracy of the energy functional

Throughout this section we fix two closed Riemannian manifolds M and M' of dimensions n and n', respectively, and denote points in them by x and y, respectively. As always we assume that the sectional curvature of M' is nonpositive. Denote by κ the smallest integer satisfying $\kappa > 2 + \frac{n+n'}{2}$ and let ζ be a homotopy class of maps from M to M'. It is a standard result (see e.g. [Pa]) that for any $k \geq \kappa$ the space $\mathcal{H}^{(k+2)}$ of maps of Sobolev class H^{k+2} from M to M' is a Hilbert manifold. Its tangent space at a smooth map $u \colon M \longrightarrow M'$ is given by the space $H^{k+2}(u^*TM')$ of H^{k+2}-sections of the pull-back u^*TM' of the tangent bundle of M'. We denote the connected component of $\mathcal{H}^{(k+2)}$ of maps in the homotopy class ζ by $\mathcal{H}_\zeta^{(k+2)}$.

Consider the energy functional

$$E(u) = \frac{1}{2} \int_M \|d_x u\|^2 \,\mathrm{dvol}(x) .$$

Under our assumption on the curvature of M', for any homotopy class ζ of maps from M to M' the critical set of the restriction of E to ζ is the nonempty, connected, compact set of minima [ES, Ha, SY], which by elliptic regularity consists of smooth maps.

For $k \geq \kappa$ as above and $u \in \mathcal{H}^{(k+2)}$, we consider the scale of Hilbert spaces

$$T_u\mathcal{H}^{(k+2)} \cong H^{k+2}(u^*TM') \hookrightarrow H^k(u^*TM') \hookrightarrow L^2(u^*TM') .$$

The above embeddings are dense, and we have $d_uE = -\tau(u) \in H^k(u^*TM')$, where E is now considered as a functional on $\mathcal{H}^{(k+2)}$. Furthermore, the operator $\nabla\tau(u)\colon H^{k+2}(u^*TM') \longrightarrow H^k(u^*TM')$ is the restriction of an unbounded selfadjoint elliptic operator on $L^2(u^*TM')$ with domain $H^2(u^*TM')$ (for an explicit formula, compare Lemma 2.1 below). In particular, it has discrete, nonnegative spectrum.

The goal of this section is to describe sufficient conditions on the homotopy class ζ of maps from M to M' which ensure that the restriction $E|_\zeta$ of the energy functional to $\mathcal{H}_\zeta^{(k+2)}$ has a nondegenerate critical set for all $k \geq \kappa$ meaning that $Crit(E|_\zeta) \subset \mathcal{H}_\zeta^{(k+2)}$ is a closed C^2-submanifold such that for any $u \in Crit(E|_\zeta)$ we have

$$T_uCrit(E|_\zeta) = \operatorname{Ker}\nabla\tau(u)\ (= \operatorname{Coker}\nabla\tau(u))\,.$$

We start by computing the second variation of the energy functional. Given a 2-parameter family of maps $u_{s,t}\colon M \longrightarrow M'$ in $\mathcal{H}^{(\kappa+2)}$, we write

$$\frac{\partial u_{s,t}}{\partial s}\bigg|_{s=0} = \varphi_t \quad\text{and}\quad \frac{\partial u_{s,t}}{\partial t}\bigg|_{t=0} = \psi_s\,.$$

By the definition of the tension field τ we have

$$\frac{\partial E(u_{s,t})}{\partial t}\bigg|_{t=0} = -\int_M \langle \tau(u_{s,0}), \psi_s\rangle\, \mathrm{dvol}(x)\,.$$

Hence for the second variation we obtain

$$\begin{aligned}\frac{\partial^2 E(u_{s,t})}{\partial s\partial t}\bigg|_{(s,t)=(0,0)} &= -\int_M \frac{\partial}{\partial s}\langle \tau(u_{s,0}), \psi_s\rangle|_{s=0}\ \mathrm{dvol}(x)\\ &= -\int_M \langle \nabla\tau(u_{0,0})\cdot\varphi_0, \psi_0\rangle + \langle \tau(u_{0,0}), (\nabla\psi_0)\cdot\varphi_0\rangle\ \mathrm{dvol}(x)\,.\end{aligned}$$

Note that if $u = u_{0,0}$ is harmonic, i.e. $\tau(u) = 0$, the second summand in the last equation vanishes. We will have occasion to use the following coordinate description of the Hessian of the energy at a harmonic map.

LEMMA 2.1. *Let $u\colon M \longrightarrow M'$ be a harmonic map between closed Riemannian manifolds. Given two sections $\varphi, \psi \in C^\infty(u^*TM')$, the Hessian $H_u(\varphi,\psi)$ is of the form $\int_M h_u(\varphi,\psi)\,\mathrm{dvol}(x)$, where in local coordinates $x^1,\dots,x^n$ of M, the density h_u is given by*

$$h_u(\varphi,\psi) = g^{ij}\langle \nabla_{X_i}\varphi, \nabla_{X_j}\psi\rangle - g^{ij}\langle R'(\frac{\partial u}{\partial x^i},\varphi)\psi, \frac{\partial u}{\partial x^j}\rangle\,. \tag{2.1}$$

Here R' denotes the curvature tensor of M', $X_i = \frac{\partial}{\partial x^i}$, ∇ denotes the covariant derivative along u, and g^{ij} is the inverse of the metric tensor g on M. Equivalently, the quadratic form H_u can be represented as

$$H_u(\varphi, \psi) = - \int_M \langle J_u \varphi, \psi \rangle \, \mathrm{dvol}(x) \, ,$$

where for any $v \in \mathcal{H}^{(\kappa+2)}$ the operator $J_v = \nabla \tau(v)$ on $L^2(v^* TM')$ is selfadjoint and elliptic and is given in local coordinates by

$$J_v \varphi = \frac{1}{\sqrt{g}} \nabla_{X_i} (\sqrt{g} g^{ij} \nabla_{X_j} \varphi) - g^{ij} R'(\frac{\partial v}{\partial x^i}, \varphi) \frac{\partial v}{\partial x^j} \, . \quad \square \tag{2.2}$$

The proof of this lemma is a standard computation (see e.g. Theorem 8.6.1 in [Jo]). Recall that, due to elliptic regularity theory, for any $u \in Crit(E)$ one has $Null(J_u) \subset C^\infty(u^* TM')$. Lemma 2.1 then yields the following well-known result.

COROLLARY 2.2. *Assume that M' has nonpositive sectional curvature. Then given a harmonic map $u \colon M \longrightarrow M'$, in normal coordinates $x^1, \dots, x^n$ at any given $x \in M$, every section $\varphi \in Null(J_u)$ satisfies*

$$\nabla_{X_i} \varphi = 0 \tag{2.3}$$

and

$$\langle R'(\frac{\partial u}{\partial x^i}, \varphi) \varphi, \frac{\partial u}{\partial x^i} \rangle = 0 \tag{2.4}$$

for $1 \leq i \leq n$.

Proof. According to (2.1), in normal coordinates at a given point $x \in M$ and for $\varphi = \psi \in Null(J_u)$ we have

$$0 = \sum_{i=1}^n \langle \nabla_{X_i} \varphi, \nabla_{X_i} \varphi \rangle + \sum_{i=1}^n - \langle R'(\frac{\partial u}{\partial x^i}, \varphi) \varphi, \frac{\partial u}{\partial x^i} \rangle . \tag{2.5}$$

As M' has nonpositive sectional curvature, each of the terms in (2.5) is non-negative and thus all have to vanish individually. $\square$

We next obtain a useful characterization of the homotopy classes for which the restriction of the energy is nondegenerate in our sense.

COROLLARY 2.3. *Assume that M' has nonpositive sectional curvature. Then the restriction $E|_\zeta$ of the energy functional to $\mathcal{H}_\zeta^{(\kappa+2)}$ is nondegenerate if and only if the set of its critical points $Crit(E|_\zeta)$ is a closed finite-dimensional C^2-submanifold of $\mathcal{H}_\zeta^{(\kappa+2)}$ and for any $u \in Crit(E|_\zeta)$ we have*

$$\dim(Null(J_u)) \leq \dim Crit(E|_\zeta) . \tag{2.6}$$

REMARK 2.4. Note that if $Crit(E|_\zeta) \subset \mathcal{H}^{(\kappa+2)}$ is a C^2- submanifold, one has

$$Null(J_u) \supseteq T_u Crit(E|_\zeta) \tag{2.7}$$

for any $u \in Crit(E|_\zeta)$. Thus inequality (2.6) implies equality in (2.7).

Proof of Corollary 2.3. Clearly, the two stated conditions are necessary for $E|_\zeta$ to be nondegenerate. To see that they are also sufficient, note that for any $u \in Crit(E|_\zeta)$ the operator J_u is self-adjoint and has discrete spectrum. Hence $Null(J_u)^\perp \cap C^\infty(u^*TM')$ is an invariant subspace for J_u, where $Null(J_u)^\perp$ denotes the orthogonal complement with respect to the L^2 inner product defined on the space of L^2-sections of u^*TM'. It then follows that J_u is non-degenerate if $Null(J_u) = T_u Crit(E|_\zeta)$. In view of the above remark, this holds if $\dim Null(J_u) \leq \dim Crit(E|_\zeta)$. □

REMARK 2.5. Using elliptic regularity theory and the selfadjointness of J_u one verifies that for any homotopy class ζ and for any $k \geq \kappa$ the restriction of the energy E to $\mathcal{H}_\zeta^{(k+2)}$ is nondegenerate if and only if the restriction of E to $\mathcal{H}_\zeta^{(\kappa+2)}$ is nondegenerate.

We now apply the criterion of Corollary 2.3 to several families of homotopy classes of maps.

PROPOSITION 2.6. *Assume that M' has nonpositive sectional curvature and that ζ is the trivial homotopy class of maps from M to M', where M is any closed Riemannian manifold. Then the restriction $E|_\zeta$ of E to $\mathcal{H}_\zeta^{(\kappa+2)}$ is nondegenerate.*

Proof. According to Hartman [Ha], zero is the only critical value of the restriction of the energy functional to the trivial homotopy class. The corresponding critical set consists precisely of the constant maps, so that $Crit(E|_\zeta)$ is a C^2-submanifold of $\mathcal{H}_\zeta^{(\kappa+2)}$ diffeomorphic to M'.

Let $u\colon M \longrightarrow M'$ be any constant map. According to Corollary 2.3, it remains to show that $\dim Null(J_u) \leq \dim M'$. Corollary 2.2 implies that any $\varphi \in Null(J_u)$ satisfies $\nabla_{X_i}\varphi \equiv 0$ for all $1 \leq i \leq n$. This means that φ is a parallel section of $u^*TM' \cong M \times T_yM'$, where $y = u(M)$. As a parallel section is determined by its value at one point, the inequality $\dim Null(J_u) \leq \dim M'$ follows. □

PROPOSITION 2.7. *Assume that M' is a flat torus $\mathbf{R}^{n'}/\Lambda$, i.e. the quotient of flat $\mathbf{R}^{n'}$ by a lattice $\Lambda \subset \mathbf{R}^{n'}$ of maximal rank, and that M is any closed Riemannian manifold. Then, for any homotopy class ζ of maps from M to M', the restriction $E|_\zeta$ of the energy functional to $\mathcal{H}_\zeta^{(\kappa+2)}$ is nondegenerate.*

Proof. By the work of Schoen and Yau [SY], $Crit(E|_\zeta)$ is a compact connected manifold, possibly with (Lipschitz) boundary, whenever the target space M' has nonpositive sectional curvature, and this manifold is immersed into M' by the evaluation map at a point. As the isometry group of any n'–dimensional flat torus contains the n'-dimensional subgroup of translations, we see that for any homotopy class ζ any $u \in Crit(E|_\zeta)$ is an interior point and we have $\dim T_u Crit(E|_\zeta) \geq n'$. In particular, $Crit(E|_\zeta)$ is a compact manifold without boundary of dimension n'. On the other hand, Corollary 2.2 again implies that $\varphi \in Null(J_u)$ satisfies $\nabla_{X_i}\varphi \equiv 0$ for all $1 \leq i \leq n$, which means that φ is a parallel section of the trivial bundle u^*TM'. Hence, as in the proof of Proposition 2.6, $\dim Null(J_u) \leq n'$, which together with Corollary 2.3 proves the claim. □

PROPOSITION 2.8. *Assume that M' has negative sectional curvature. Then for any homotopy class ζ of maps from any closed Riemannian manifold M to M' the restriction $E|_\zeta$ of the energy functional to $\mathcal{H}_\zeta^{(\kappa+2)}$ is nondegenerate.*

Proof. For $\dim M = 0$ the result is trivial, so we may assume $\dim M \geq 1$. Furthermore, for any homotopy class ζ, $Crit(E|_\zeta)$ is a closed C^2-submanifold of $\mathcal{H}_\zeta^{(\kappa+2)}$ by [Ha].

So fix a non-trivial homotopy class ζ (the trivial homotopy class was already covered in Proposition 2.6) and a harmonic map $u \in Crit(E|_\zeta)$. According to equation (2.3) of Corollary 2.2, in normal coordinates at a given point $x \in M$, for any $\varphi \in Null(J_u)$ and all $1 \leq i \leq n$ we have

$$\frac{\partial}{\partial x^i}\langle \varphi(x), \varphi(x)\rangle = 2\langle \nabla_{X_i}\varphi(x), \varphi(x)\rangle = 0\,.$$

This shows that $\|\varphi(x)\|$ is a constant function on M. By equation (2.4) we also know that

$$(2.8) \qquad \langle R'(\frac{\partial u}{\partial x^i}, \varphi)\varphi, \frac{\partial u}{\partial x^i}\rangle = 0$$

for all $1 \leq i \leq n$. We now consider two cases.

CASE 1. Suppose there exist a map $u_0 \in Crit(E|_\zeta)$ and a point $x_0 \in M$ with $\dim u_{0*}(T_{x_0}M) \geq 2$. Since M' has negative sectional curvature, we conclude from equation (2.8) considered at the point x_0 and with $u = u_0$ that any $\varphi \in Null(J_{u_0})$ satisfies $\varphi(x_0) = 0$. Hence, as $||\varphi(x)||$ is a constant function on M, we conclude that $Null(J_{u_0}) = \{0\}$. On the other hand, Hartman [Ha, Cor. to Thm. H] showed that u_0 is the only harmonic map in this homotopy class. Now Corollary 2.3 proves the claim in this first case.

CASE 2. Suppose that for all $u \in Crit(E|_\zeta)$ we have $\operatorname{rank} u_* \leq 1$ everywhere on M. By Proposition 2.9 below, u can be written as $u = \gamma \circ g$, where $g\colon M \longrightarrow S^1$ is smooth and $\gamma\colon S^1 \longrightarrow M'$ is a parametrization of a closed geodesic proportional to arc length. Rotating the geodesic gives a 1-parameter family $u_s(x) = \gamma(g(x) + s)$ of harmonic maps homotopic to u, so that $\dim T_u Crit(E|_\zeta) \geq 1$. On the other hand, any $\varphi \in Null(J_u)$ satisfies $||\varphi|| = const$, and equation (2.8) yields $\varphi(x) \in u_*(T_xM)$ for the open dense set of points $x \in M$ where $\operatorname{rank} u_* = 1$. Thus we find $\dim Null(J_u) \leq 1$, so that the conclusion of the proposition again follows from Corollary 2.3. □

To finish the proof of the last proposition, we need the following result (cf. also assertion (I) in [Ha]).

PROPOSITION 2.9. *Assume that $u\colon M \longrightarrow M'$ is a nonconstant harmonic map with* $\operatorname{rank} u_* \leq 1$ *everywhere on M. Then there exist a closed geodesic in M' with parametrization $\gamma\colon S^1 \longrightarrow M'$ proportional to arc length and a smooth surjective map $g\colon M \longrightarrow S^1$ such that $u = \gamma \circ g$.*

Proof. Sampson [Sam, Theorem 3] proved that the image of u coincides with the image of a closed geodesic. There is no problem in defining g when this geodesic has no self-intersections. In the general case, we fix a degree 1 parametrization $\gamma\colon S^1 \longrightarrow M'$ of the geodesic proportional to arc length. Note that g can be continuously defined on

$$M_{reg} = \{x \in M \mid \operatorname{rank} d_xu = 1\},$$

because for points $x \in M_{reg}$ the value $u(x)$ together with the tangent line $d_xu(T_xM) \subset T_{u(x)}M'$ uniquely determine the parameter $s \in S^1$ with

$$\gamma(s) = u(x)$$
$$\dot\gamma(s) \in d_x u(T_x M).$$

Clearly, g can also be continuously extended to those $x \in M_{sing} = M \setminus M_{reg}$ which map to simple points of $\gamma(S^1)$, as there we only have one choice for $g(x)$. It remains to define g at points $x_0 \in M_{sing}$ for which $u(x_0)$ is a crossing point of the geodesic γ, i.e. there are distinct points $s_1, \dots, s_k$ in S^1, $k \geq 2$, so that

$$\gamma^{-1}(u(x_0)) = \{s_1, \dots, s_k\} \subset S^1 .$$

Choose disjoint open neighborhoods $V_1, \dots, V_k$ of the s_i in S^1, and set $U_i := (g_{|M_{reg}})^{-1}(V_i)$. As M_{reg} is open and $g_{|M_{reg}}$ is continuous, the U_i are disjoint open sets in M, and if U is a small enough ball in M around x_0, then the U_i cover $U \cap M_{reg}$. According to [Ya], the Hausdorff codimension of M_{sing} satisfies $\mathrm{codim}_H(M_{sing}) \geq 2$ for any non-constant harmonic map u, so that in particular $U \cap M_{reg}$ is connected by standard dimension theory (cf. [HW]). This implies that $U \cap M_{reg}$ is contained in one of the U_i, and so setting $g(x_0) = s_i$ makes g continuous at x_0. Thus we can extend g continuously to all of M, and as $u = \gamma \circ g$, u is smooth and γ is a local diffeomorphism, it follows that g is smooth as well. □

As a final result in this section, we prove that the set of homotopy classes for which the energy functional is nondegenerate is closed under products in the following sense.

PROPOSITION 2.10. *For $i \in \{1,2\}$, let M_i' be manifolds with nonpositive sectional curvature with dimensions n_i', and let ζ_i be homotopy classes of maps from M to M_i' such that the restrictions $E_{|\zeta_i}$ of the energy to $\mathcal{H}_{\zeta_i}^{(\kappa_i+2)}$ are nondegenerate, where κ_i is the smallest integer satisfying $\kappa_i > 2 + \frac{n+n_i'}{2}$. Then for the homotopy class $\zeta = \zeta_1 \times \zeta_2$ of maps from M to $M_1' \times M_2'$ the restriction $E|_\zeta$ of the energy functional to $\mathcal{H}_\zeta^{(\kappa+2)}$ is also nondegenerate, where κ is the smallest integer satisfying $\kappa > 2 + \frac{n+n_1'+n_2'}{2}$.*

Proof. Maps $u\colon M \longrightarrow M_1' \times M_2'$ in $\mathcal{H}_\zeta^{(\kappa+2)}$ are in one-to-one correspondence with pairs $(u_1, u_2) \in \mathcal{H}_{\zeta_1}^{(\kappa+2)} \times \mathcal{H}_{\zeta_2}^{(\kappa+2)}$. A trivial calculation shows that $E(u) = E(u_1) + E(u_2)$, from which it is clear that

$$Crit(E|_\zeta) = Crit(E_{|\zeta_1}) \times Crit(E_{|\zeta_2}) .$$

As $u^*T(M_1' \times M_2') = u_1^*TM_1' \oplus u_2^*TM_2'$, any $\varphi \in Null(J_u)$ splits as $\varphi = \varphi_1 + \varphi_2$ with $\varphi_i \in Null(J_{u_i})$, and so the claim follows from Corollary 2.3 and Remark 2.5. □

3. Fredholm argument

There is a general setting for studying maps which satisfy some elliptic systems of partial differential equations. The set of solutions to the equations can often be expressed as the zero locus of some section in a Hilbert or Banach bundle over a manifold of maps in a considered class. If this section is transverse to the zero section of the bundle, the space of solutions is a manifold. However, for many interesting problems, this section is not transverse to the zero section, so that the solution set is not as well behaved. In these cases one wishes to find suitable perturbations of the original elliptic system for which the solution space still is a manifold of the expected dimension. Then one is faced with the problem of proving that the essential properties of the solution set of the generically perturbed equation, such as its cobordism class, are as independent as possible of the choice of perturbation.

In this paper, our point of view is slightly different, as we are interested in the solvability of the perturbed harmonic map equations, where the solution set in the unperturbed case is rather well understood. Nevertheless the methods outlined above apply. The purpose of this section is to give a self-contained account of a fairly general argument, which is sufficient for our purposes but at the same time general enough to be useful in other contexts. In §5 we verify that the assumptions made here are satisfied in our setting.

Let $\mathcal{H}$ be a Hilbert manifold, possibly with boundary, and $\mathcal{B}$ a Banach manifold without boundary, both of class C^r with $r \geq 1$. We denote elements of $\mathcal{H}$ and $\mathcal{B}$ by x and y, respectively. Let $\mathcal{E} \longrightarrow \mathcal{H} \times \mathcal{B}$ be a Hilbert space bundle of class C^r, endowed with a connection ∇. We will use $\mathcal{E}_{x,y}$ for the fibre of the bundle $\mathcal{E}$ above $(x,y) \in \mathcal{H} \times \mathcal{B}$, and $\nabla^{\mathcal{H}}s$ and $\nabla^{\mathcal{B}}s$ for the restrictions of ∇s to $T\mathcal{H}$ and $T\mathcal{B}$, respectively, where $s\colon \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{E}$ is a section.

Let $s\colon \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{E}$ be a C^l-section with $1 \leq l \leq r$. We say that s has property $(\mathcal{R})$ if

$$(\mathcal{R}) \qquad (\nabla^{\mathcal{B}}s)_{x,y}\colon T_y\mathcal{B} \longrightarrow \mathcal{E}_{x,y} \text{ has a right inverse for all } (x,y) \in s^{-1}(0)\,.$$

Similarly, we say that the section s has the Fredholm property $(\mathcal{F}_l)$ if

$(\mathcal{F}_i)$ $(\nabla^{\mathcal{H}} s)_{x,y}\colon T_x\mathcal{H} \longrightarrow \mathcal{E}_{x,y}$ is Fredholm of index i for all $(x,y) \in \mathcal{H} \times \mathcal{B}$.

A section s with property $(\mathcal{F}_i)$ is said to have property $(\mathcal{O})$ if the determinant line bundle

$(\mathcal{O})$ $\quad \mathcal{D}et(\nabla^{\mathcal{H}} s) \longrightarrow \mathcal{H} \times \mathcal{B}$ is a trivial line bundle.

A detailed discussion of the determinant line bundle of a Fredholm bundle map appears in Appendix A. We record the first implications of these conditions in

PROPOSITION 3.1. *Let $s\colon \mathcal{H}\times\mathcal{B} \longrightarrow \mathcal{E}$ be a C^l-section of the C^r-Hilbert space bundle $\mathcal{E}$ with $1 \le l \le r$ as above.*

1. *If s satisfies $(\mathcal{R})$, then $s^{-1}(0)$ is a C^l-Banach submanifold of $\mathcal{H}\times\mathcal{B}$ with boundary $s^{-1}(0)\cap(\partial\mathcal{H}\times\mathcal{B})$, whose tangent space at a point $(x,y) \in s^{-1}(0)$ is $T_{x,y}s^{-1}(0) = \mathrm{Ker}\,(\nabla s)_{x,y}$.*
2. *If, in addition, s satisfies $(\mathcal{F}_i)$, then the restriction $\pi\colon s^{-1}(0) \longrightarrow \mathcal{B}$ of the projection of $\mathcal{H}\times\mathcal{B}$ to the second factor is a Fredholm map with* index π = index $\nabla^{\mathcal{H}} s$. *Moreover, $y \in \mathcal{B}$ is a regular value of π if and only if $(\nabla^{\mathcal{H}} s)_{x,y}$ is onto for all $(x,y) \in s^{-1}(0)$.*
3. *Under the assumptions of (2), for any regular value $y \in \mathcal{B}$ of π and $\pi_{|\partial s^{-1}(0)}$, the set $\pi^{-1}(y) \subset s^{-1}(0)$ is a manifold of class C^l with boundary $\pi^{-1}(y)\cap\partial s^{-1}(0)$ and dimension* $\dim \pi^{-1}(y) = \mathrm{index}\ \nabla^{\mathcal{H}} s$.
4. *If, in addition to the assumptions in (2), the section s satisfies $(\mathcal{O})$, then a choice of trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$ over $s^{-1}(0)$ determines an orientation of $\pi^{-1}(y)$ for every regular value $y \in \mathcal{B}$ of π.*

Proof. (1) The existence of a right inverse for $(\nabla^{\mathcal{B}} s)_{x,y}$ whenever $(x,y) \in s^{-1}(0)$ guarantees that $(\nabla s)_{x,y}$, as well as its restriction to $T_{x,y}(\partial\mathcal{H}\times\mathcal{B})$ for points $(x,y) \in s^{-1}(0)\cap(\partial\mathcal{H}\times\mathcal{B})$, have right inverses. The result now follows from a standard application of the implicit function theorem (cf. [La]).

(2) We need to show that the differential of π at any point of $s^{-1}(0)$ is Fredholm of the claimed index. So let $(x,y) \in s^{-1}(0)$ be fixed. We first want to describe $T_{x,y}s^{-1}(0)$ more explicitly. Denoting by $R\colon \mathcal{E}_{x,y} \longrightarrow T_y\mathcal{B}$ the right inverse of $\nabla^{\mathcal{B}} s$ (here and below we suppress the base point (x,y) from the notation for the maps involved), we split

$$T_x\mathcal{H} = H \oplus H^{\perp},$$

where $H = \mathrm{Ker}\,\nabla^{\mathcal{H}} s$, and

$$T_y\mathcal{B} = B \oplus B^c,$$

where $B = R((\operatorname{Im}\nabla^{\mathcal{H}}s)^{\perp})$, and B^c denotes a closed complement of B. The latter exists since B is finite dimensional. Now note that $T_{x,y}s^{-1}(0) = \operatorname{Ker}\nabla s$ consists of elements $(\eta, \xi) \in T_x\mathcal{H} \times T_y\mathcal{B}$ satisfying

$$\nabla^{\mathcal{H}}s \cdot \eta + \nabla^{\mathcal{B}}s \cdot \xi = 0\,. \tag{3.1}$$

We see that (3.1) has no solution if $\xi \in B \setminus \{0\}$. On the other hand, writing $\eta = \eta' \oplus \eta''$ and $\xi = \xi' \oplus \xi''$ according to the splittings above, any $\xi'' \in B^c$ determines unique elements $\eta'' \in H^{\perp}$ and $\xi' \in B$ by

$$\nabla^{\mathcal{H}}s \cdot \eta'' = -P \circ \nabla^{\mathcal{B}}s \cdot \xi''$$

and

$$\xi' = -R \circ P^{\perp} \circ \nabla^{\mathcal{B}}s \cdot \xi''\,,$$

where $P\colon \mathcal{E}_{x,y} \longrightarrow \operatorname{Im}\nabla^{\mathcal{H}}s$ and $P^{\perp}\colon \mathcal{E}_{x,y} \longrightarrow (\operatorname{Im}\nabla^{\mathcal{H}}s)^{\perp}$ are the orthogonal projections. Hence given $(\eta', \xi'') \in H \oplus B^c$, the element

$$(\eta' + \eta''(\xi''), \xi'(\xi'') + \xi'') \in T_x\mathcal{H} \times T_y\mathcal{B}$$

solves (3.1). Moreover, one can identify B^c with a complement of $H \times \{0\}$ in $T_{x,y}s^{-1}(0)$, so that

$$T_{x,y}s^{-1}(0) \cong H \oplus B^c\,. \tag{3.2}$$

Using this identification one reads off the kernel and cokernel of the differential $d\pi\colon T_{x,y}s^{-1}(0) \longrightarrow T_y\mathcal{B} = B \oplus B^c$ as

$$\operatorname{Ker} d\pi = H\,, \qquad \operatorname{Coker} d\pi \cong B\,.$$

By assumption $(\mathcal{F}_l)$, both $H = \operatorname{Ker}\nabla^{\mathcal{H}}s$ and $B = R((\operatorname{Im}\nabla^{\mathcal{H}}s)^{\perp})$ are finite dimensional. Since $R\colon (\operatorname{Im}\nabla^{\mathcal{H}}s)^{\perp} \longrightarrow B$ is in fact a linear isomorphism, we find that $B \cong \operatorname{Coker}(\nabla^{\mathcal{H}}s)_{x,y}$ and so $d\pi$ is a Fredholm operator of the same index as $\nabla^{\mathcal{H}}s$, which was to be proven. Clearly $y \in \mathcal{B}$ is a regular value of π if and only if $B = \{0\}$ for all $(x, y) \in \pi^{-1}(y)$, which in turn is equivalent to $(\nabla^{\mathcal{H}}s)_{x,y}$ being surjective for all $(x, y) \in s^{-1}(0)$.

(3) That $\pi^{-1}(y)$ is a manifold of class C^l with boundary $\pi^{-1}(y) \cap \partial s^{-1}(0)$ and tangent space $\operatorname{Ker} d\pi$ is again a standard corollary of the implicit function theorem. Since we are at a regular value of π, the cokernel vanishes and thus the dimension of the kernel equals the index of the differential $d\pi$, which was computed in (2).

(4) In the proof of (2) we saw that, given any regular value $y \in \mathcal{B}$ of π, we have $\operatorname{Ker}(\nabla^{\mathcal{H}}s)_{x,y} = \operatorname{Ker} d\pi_{x,y}$ and $\operatorname{Coker}(\nabla^{\mathcal{H}}s)_{x,y} = \{0\}$ for any $(x, y) \in \pi^{-1}(y)$. It then follows from the definition of the determinant line bundle that

$$\mathcal{D}et(\nabla^{\mathcal{H}} s)_{|\pi^{-1}(y)} = \Lambda^{\max}(\operatorname{Ker} d\pi)_{|\pi^{-1}(y)} = \Lambda^{\max}(T\pi^{-1}(y)).$$

Thus a choice of trivialization of the determinant line bundle of $\nabla^{\mathcal{H}} s$ gives rise to an orientation of the manifold $\pi^{-1}(y)$. □

In a first application of Proposition 3.1, we think of $\mathcal{H}$ as the general space of maps on which we study our system of equations and $\mathcal{B}$ as the space of perturbations. The perturbed equation is then written as $s = 0$. In this context, $(\mathcal{R})$ says that we have chosen the perturbations sufficiently general and $(\mathcal{F}_i)$ just says that the perturbed systems have a Fredholm linearization. For the third part of Proposition 3.1 to be useful, we need the additional assumption

$$(\mathcal{P}) \qquad \pi : s^{-1}(0) \longrightarrow \mathcal{B} \text{ is proper}.$$

It ensures in particular that the inverse image $\pi^{-1}(y)$ of any $y \in \mathcal{B}$ is compact.

Our next goal is to show that under the assumptions $(\mathcal{R})$, $(\mathcal{F}_i)$, $(\mathcal{O})$ and $(\mathcal{P})$ the oriented cobordism class of the manifold $\pi^{-1}(y)$ depends only on the path component of $\mathcal{B}$ in which the regular value y lies. This means that, given any two regular values y_0 and y_1 of π in the same path component, we have to construct a compact oriented manifold of dimension $i+1$ whose oriented boundary is given by the difference of the oriented manifolds $\pi^{-1}(y_0)$ and $\pi^{-1}(y_1)$. To this end, we will apply Proposition 3.1 in a slightly different setting. Namely, let two regular values $y_0, y_1 \in \mathcal{B}$ of π be given. As π is proper, the set of regular values is open, so that we can find open disks U_0 and U_1 containing y_0 and y_1 and consisting of regular values. We introduce the product Hilbert manifold $\widetilde{\mathcal{H}} = \mathcal{H} \times [0,1]$ and the Banach manifold without boundary

$$\widetilde{\mathcal{B}} := \{\gamma \in C^l([0,1], \mathcal{B}) \mid \gamma(0) \in U_0, \gamma(1) \in U_1\}$$

where $1 \leq l \leq r$ as before. The property that the paths in $\widetilde{\mathcal{B}}$ need not have fixed endpoints will turn out to be convenient later. Note that if y_0 and y_1 are in the same path component of $\mathcal{B}$, then $\widetilde{\mathcal{B}}$ is non-empty. There is a canonical map

$$\begin{array}{ccc} \widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}} & \longrightarrow & \mathcal{H} \times \mathcal{B} \\ (x, t, \gamma) & \mapsto & (x, \gamma(t)). \end{array}$$

We denote by $\widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}$ the pull-back of $\mathcal{E}$ under this map, with fiber $\widetilde{\mathcal{E}}_{x,t,\gamma} \equiv \mathcal{E}_{x,\gamma(t)}$ at the point $(x, t, \gamma) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}$. It is endowed with a connection $\widetilde{\nabla}$ induced from ∇. Let $\tilde{s}$ be the pull-back of the section s, i.e. $\tilde{s}(x, t, \gamma) = s(x, \gamma(t))$, which is again of class C^l by our choice of $\widetilde{\mathcal{B}}$. We denote by $\widetilde{\nabla}^{\widetilde{\mathcal{H}}} \tilde{s}$ and $\widetilde{\nabla}^{\widetilde{\mathcal{B}}} \tilde{s}$ the restrictions of $\widetilde{\nabla} \tilde{s}$ to $T\widetilde{\mathcal{H}}$ and $T\widetilde{\mathcal{B}}$ respectively.

Finally, we denote the restriction to $\tilde{s}^{-1}(0)$ of the projection of $\widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}$ to $\widetilde{\mathcal{B}}$ by $\tilde{\pi}$. The following proposition shows that $\tilde{s}$ inherits various properties from s.

PROPOSITION 3.2. *In the situation described above, properties* $(\mathcal{R}), (\mathcal{F}_i)$ *and* $(\mathcal{O})$ *for* s *imply properties* $(\mathcal{R})$, $(\mathcal{F}_{i+1})$ *and* $(\mathcal{O})$ *for* $\tilde{s}$, *respectively. Property* $(\mathcal{P})$ *for* π *implies property* $(\mathcal{P})$ *for* $\tilde{\pi}$.

Proof. Fix $(x,t,\gamma) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}$ and observe that for $(\eta, \theta) \in T_x\mathcal{H} \times T_t[0,1]$ we have

$$(\widetilde{\nabla}^{\widetilde{\mathcal{H}}}\tilde{s})_{x,t,\gamma} \cdot (\eta, \theta) = \nabla^{\mathcal{H}} s \cdot \eta + \theta \nabla^{\mathcal{B}} s \cdot \dot{\gamma} \in \mathcal{E}_{x,\gamma(t)} \, .$$

In particular, $(\widetilde{\nabla}^{\widetilde{\mathcal{H}}}\tilde{s})_{x,t,\gamma}$ is an extension of $(\nabla^{\mathcal{H}} s)_{x,\gamma(t)}$ in the sense of Lemma A.1 and Corollary A.6, from which it follows that if the section s satisfies $(\mathcal{F}_i)$ (resp. $(\mathcal{F}_i)$ and $(\mathcal{O})$), then $\tilde{s}$ satisfies $(\mathcal{F}_{i+1})$ (resp. $(\mathcal{F}_{i+1})$ and $(\mathcal{O})$).

Now we turn to the construction of a right inverse. Again fix $(x,t,\gamma) \in \tilde{s}^{-1}(0)$. As s is assumed to have property $(\mathcal{R})$, there exists a right inverse $R_{x,\gamma(t)} \colon \mathcal{E}_{x,\gamma(t)} \longrightarrow T_{\gamma(t)}\mathcal{B}$ of $(\nabla^{\mathcal{B}} s)_{x,\gamma(t)}$ and we are supposed to construct a right inverse $\widetilde{R}_{x,t,\gamma} \colon \mathcal{E}_{x,\gamma(t)} \longrightarrow T_\gamma \widetilde{\mathcal{B}}$ for $(\widetilde{\nabla}^{\widetilde{\mathcal{B}}}\tilde{s})_{x,t,\gamma}$. Notice that $T_\gamma\widetilde{\mathcal{B}}$ consists of C^l-sections of the pull-back bundle $\gamma^* T\mathcal{B}$ [1]). As $[0,1]$ is contractible, we may assume that a trivialization of the form

$$\gamma^*(T\mathcal{B}) \cong [0,1] \times T_{\gamma(t)}\mathcal{B} \tag{3.3}$$

has been chosen. Using this trivialization, we identify

$$T_\gamma \widetilde{\mathcal{B}} \cong C^l([0,1], T_{\gamma(t)}\mathcal{B})$$

and define $\widetilde{R}_{x,t,\gamma} \colon \mathcal{E}_{x,\gamma(t)} \longrightarrow T_\gamma\widetilde{\mathcal{B}}$ as the constant section

$$(\widetilde{R}_{x,t,\gamma}(\mu))(t') := R_{x,\gamma(t)}(\mu) \qquad \text{for } t' \in [0,1]$$

with respect to the trivialization. One easily checks that

$$(\widetilde{\nabla}^{\widetilde{\mathcal{B}}}\tilde{s})_{x,t,\gamma} \circ \widetilde{R}_{x,t,\gamma}(\mu) = (\nabla^{\mathcal{B}} s)_{x,\gamma(t)} \circ R_{x,\gamma(t)}(\mu) = \mu \, ,$$

so $\widetilde{R}_{x,t,\gamma}$ is indeed the required inverse.

Finally, it remains to verify that property $(\mathcal{P})$ for π implies property $(\mathcal{P})$ for $\tilde{\pi}$. Given a compact subset $\widetilde{K} \subset \widetilde{\mathcal{B}}$, the corresponding subset

$$K := \{\gamma(t) \mid \gamma \in \widetilde{K}, t \in [0,1]\} \subset \mathcal{B}$$

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[1]) This would not be true if $\widetilde{\mathcal{B}}$ consisted of paths with fixed endpoints.

is also compact. So, using the assumption that π is proper, we conclude that $\pi^{-1}(K) \subset \mathcal{H} \times \mathcal{B}$ and therefore also its projection $\widehat{K}$ onto $\mathcal{H}$ are compact. Hence

$$\tilde{\pi}^{-1}(\widetilde{K}) \subset \widehat{K} \times [0,1] \times \widetilde{K}$$

is a closed subset of a compact set, which proves the claim. □

REMARK 3.3. In the proof we have seen that $\widetilde{\nabla}^{\widetilde{\mathcal{H}}}\tilde{s}$ is an extension of $\nabla^{\mathcal{H}}s$. Hence, according to Corollary A.6, we need to fix a trivialization for $T[0,1]$ in order to identify trivializations of $\mathcal{D}et(\nabla^{\mathcal{H}}s)$ and $\mathcal{D}et(\widetilde{\nabla}^{\widetilde{\mathcal{H}}}\tilde{s})$. We choose it to be the standard one given by $\frac{\partial}{\partial t}$.

We are now ready to state the main result of this section:

THEOREM 3.4. *Suppose $\mathcal{H}$ is a Hilbert manifold and $\mathcal{B}$ is a pathconnected Banach manifold, both of class C^r ($r \geq 1$) and without boundary, and suppose that $\mathcal{E} \longrightarrow \mathcal{H} \times \mathcal{B}$ is a C^r-Hilbert space bundle. Assume that $s\colon \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{E}$ is a C^l-section ($1 \leq l \leq r$) such that assumptions $(\mathcal{R}), (\mathcal{F}_i)$ and $(\mathcal{P})$ are satisfied. Then the following statements are true:*

1. *The inverse image $s^{-1}(0) \subset \mathcal{H} \times \mathcal{B}$ is a C^l-Banach submanifold without boundary.*
2. *The inverse image $\pi^{-1}(y)$ of any regular value y of the projection map $\pi\colon s^{-1}(0) \longrightarrow \mathcal{B}$ is a closed C^l-manifold of dimension $i =$ index $\nabla^{\mathcal{H}}s$.*
3. *The cobordism class of $\pi^{-1}(y)$ is independent of the choice of the regular value $y \in \mathcal{B}$.*
4. *Suppose in addition that s satisfies $(\mathcal{O})$ and that some trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}}s)$ over $s^{-1}(0)$ has been fixed. Then the manifold $\pi^{-1}(y)$ is oriented and its oriented cobordism class D is independent of the choice of the regular value $y \in \mathcal{B}$.*

Proof. Parts (1) and (2) of the statement were already proved in Proposition 3.1. To prove (3), let $y_0, y_1 \in \mathcal{B}$ be two regular values of π, and choose open disk neighborhoods U_0 and U_1 consisting of regular values as above. Our first claim is that the preimages of any two points y, y' in U_0 are cobordant (and similarly for U_1). To see this, connect y and y' by a differentiable, embedded path $\gamma\colon [0,1] \longrightarrow U_0$. As U_0 consists of regular values for π only, one can see directly that $\pi^{-1}(\gamma([0,1]))$ is diffeomorphic to the product $\pi^{-1}(y) \times [0,1]$, so statements (3) and (4) are true for all pairs of points $y, y' \in U_0$.

Due to this observation it is sufficient to produce a cobordism between the preimages under π of *some* points $y \in U_0$ and $y' \in U_1$. Here we use the construction of the manifold $\widetilde{\mathcal{B}}$ as above and combine Proposition 3.2 with Proposition 3.1 to see that any common regular value $\gamma \in \widetilde{\mathcal{B}}$ of $\widetilde{\pi}$ and $\widetilde{\pi}_{|\partial\widetilde{s}^{-1}(0)}$ gives rise to a compact oriented manifold $\widetilde{\pi}^{-1}(\gamma)$ with boundary $\widetilde{\pi}^{-1}(\gamma) \cap \partial\widetilde{s}^{-1}(0)$, of dimension $i+1$ and class C^l. The existence of a common regular value γ of $\widetilde{\pi}$ and $\widetilde{\pi}_{|\partial\widetilde{s}^{-1}(0)}$ is guaranteed by property $(\mathcal{P})$ for $\widetilde{\pi}$ (which by Proposition 3.2 follows from our assumption of property $(\mathcal{P})$ for π), since together with the Sard-Smale Theorem it implies that the sets of regular values of $\widetilde{\pi}$ and $\widetilde{\pi}_{|\partial\widetilde{s}^{-1}(0)}$ are *open* and dense. We claim that $\widetilde{\pi}^{-1}(\gamma)$ is a cobordism between $\pi^{-1}(\gamma(0))$ and $\pi^{-1}(\gamma(1))$.

To prove this claim, recall first that Proposition 3.1 shows that the boundary of $\widetilde{\pi}^{-1}(\gamma)$ is given by

$$\partial\widetilde{\pi}^{-1}(\gamma) = \widetilde{\pi}^{-1}(\gamma) \cap (\partial\widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}).$$

As $\mathcal{H}$ has no boundary, $\partial\widetilde{\mathcal{H}} = \mathcal{H} \times \partial[0,1]$ is given as a disjoint union of $\mathcal{H} \times \{0\}$ and $\mathcal{H} \times \{1\}$. Observe also that the canonical map identifies $\mathcal{H} \times \{t\} \times \{\gamma\} \subset \widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}$ with $\mathcal{H} \times \{\gamma(t)\} \subset \mathcal{H} \times \mathcal{B}$. Putting things together, we see that the boundary of $\widetilde{\pi}^{-1}(\gamma)$ is canonically identified as

$$\begin{aligned}
\widetilde{\pi}^{-1}(\gamma) \cap \partial(\widetilde{\mathcal{H}} \times \widetilde{\mathcal{B}}) &= \widetilde{s}^{-1}(0) \cap \Big(\mathcal{H} \times \{0\} \times \{\gamma\} \bigsqcup \mathcal{H} \times \{1\} \times \{\gamma\}\Big) \\
&\cong s^{-1}(0) \cap \Big(\mathcal{H} \times \{\gamma(0)\} \bigsqcup \mathcal{H} \times \{\gamma(1)\}\Big) \\
&= \pi^{-1}(\gamma(0)) \bigsqcup \pi^{-1}(\gamma(1)).
\end{aligned}$$

Finally, to prove (4), we need to consider orientations. Proposition 3.1(4) asserts that the trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$ gives rise to an orientation of $\pi^{-1}(y)$ for any regular value $y \in \mathcal{B}$. Now consider the path γ between regular values $y \in U_0$ and $y' \in U_1$ constructed in the proof of (3) above. We claim that under the additional assumption in (4), $\widetilde{\pi}^{-1}(\gamma)$ is in fact an oriented cobordism between the oriented manifolds $\pi^{-1}(y)$ and $\pi^{-1}(y')$. Recall that by Proposition 3.2 the bundle $\mathcal{D}et(\widetilde{\nabla}^{\mathcal{H}} \widetilde{s})$ over $\widetilde{s}^{-1}(0)$ inherits a trivialization from those of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$ and $T[0,1]$, which in turn gives rise to an orientation of $\widetilde{\pi}^{-1}(\gamma)$. The latter induces an orientation on $\partial\widetilde{\pi}^{-1}(\gamma)$ in the usual way, meaning that a positively oriented basis for the boundary is one which by adding an outward normal vector in the last slot turns into a positively oriented basis according to the given orientation.

Notice that the projection of

$$\widetilde{\pi}^{-1}(\gamma) = \{(x,t,\gamma) \mid (x,\gamma(t)) \in s^{-1}(0)\}$$

to $[0,1]$ is a submersion near $t=0$ and $t=1$, as γ maps neighborhoods of the boundary of $[0,1]$ into the open sets U_0 and U_1 consisting of regular values for π. It follows that we can identify the normal direction to the boundary of $\tilde{\pi}^{-1}(\gamma)$ with the $\frac{\partial}{\partial t}$-direction via this projection. Now at the right end, $t = 1$, $\frac{\partial}{\partial t}$ is an outward pointing normal, so by Remark 3.3 the orientation for $\pi^{-1}(y')$ as a boundary piece of $\tilde{\pi}^{-1}(\gamma)$ agrees with its orientation coming from the trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$. Conversely, at the left end, $t = 0$, $\frac{\partial}{\partial t}$ is an inward pointing normal, so the orientation for $\pi^{-1}(y)$ as a boundary piece of $\tilde{\pi}^{-1}(\gamma)$ is opposite to its orientation coming from the trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$. In other words, we see that, as oriented manifolds,

$$\partial \tilde{\pi}^{-1}(\gamma) = \pi^{-1}(y') - \pi^{-1}(y),$$

which was to be proven. □

COROLLARY 3.5. *If under the assumptions of Theorem 3.4 the cobordism class D is nontrivial, then $\pi^{-1}(y) \neq \varnothing$ for all $y \in \mathcal{B}$.*

Proof. The assertion is true for the dense set of regular values of π. If $y \in \mathcal{B}$ is arbitrary, let $y_i \longrightarrow y$ be a sequence of regular values and $m_i \in \pi^{-1}(y_i)$ a sequence of preimages. As $\{y_i\} \cup \{y\}$ is compact and π is proper, a subsequence of the m_i converges to some $m \in \pi^{-1}(y)$. □

The following example serves to illustrate the results of this section in a very simple case.

EXAMPLE 3.6. Let $\mathcal{H}$ be a finite-dimensional, closed, not necessarily orientable Riemannian manifold, and let $\mathcal{B}$ be the space of vector fields on $\mathcal{H}$ (of some prescribed Sobolev class which is inessential here). We denote the pull back of $T\mathcal{H}$ over $\mathcal{H} \times \mathcal{B}$ by $\mathcal{E}$. The bundle $\mathcal{E}$ admits the tautological section $\sigma \colon \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{E}$ given by

$$\sigma(x, V) := V(x) \in \mathcal{E}_{x,V} = T_x\mathcal{H}.$$

It is easy to see that σ satisfies all the assumptions of Theorem 3.4. The map $(\nabla^{\mathcal{B}}\sigma)_{x,V}$ is obviously surjective, so we can always find a right inverse, which means that σ has property $(\mathcal{R})$. As $(\nabla^{\mathcal{H}}\sigma)_{x,V} = (\nabla V)_x$ is the ordinary covariant derivative of the vector field V, which is a linear endomorphism of the finite dimensional vector space $T_x\mathcal{H}$, it is clearly Fredholm of index 0, so σ has property $(\mathcal{F}_0)$. Property $(\mathcal{P})$ for σ follows from the compactness of $\mathcal{H}$. Finally, $\mathcal{H} \times \mathcal{B}$ has $\mathcal{H} \times \{0\}$ as a deformation retract, and we have, canonically,

$$\mathcal{D}et(\nabla^{\mathcal{H}}\sigma)_{|\mathcal{H}\times\{0\}} = \Lambda^{\max} T\mathcal{H} \otimes \Lambda^{\max} T\mathcal{H} \simeq \underline{\mathbf{R}},$$

where $\underline{\mathbf{R}}$ denotes the trivial line bundle over $\mathcal{H} \times \{0\}$ and the bundle isomorphism uses the inner product. Thus σ also satisfies condition $(\mathcal{O})$. To fix a trivialization, we define a section $\mu_0 \colon \mathcal{H} \times \{0\} \longrightarrow \mathcal{D}et(\nabla^{\mathcal{H}}\sigma)_{|\mathcal{H}\times\{0\}}$ by setting

$$\mu_0(x) = (e_1 \wedge \cdots \wedge e_n) \otimes (e_1 \wedge \cdots \wedge e_n) \in \Lambda^{\max} T_x\mathcal{H} \otimes \Lambda^{\max} T_x\mathcal{H} \,,$$

which is clearly independent of the choice of the orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x\mathcal{H}$ and varies smoothly with $x \in \mathcal{H}$.

Notice that $V \in \mathcal{B}$ is a regular value for the projection

$$\pi \colon \sigma^{-1}(0) \longrightarrow \mathcal{B}$$

if and only if the vector field V is transverse to the zero section of $T\mathcal{H}$. This is equivalent to the fact that $(\nabla V)_x$ is a linear isomorphism of $T_x\mathcal{H}$ whenever $V(x) = 0$. According to Theorem 3.4, a choice of trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}}\sigma)$ gives rise to an orientation of the 0-dimensional manifold $V^{-1}(0)$, i.e. an element $\operatorname{sgn} x \in \{\pm 1\}$ for each $x \in V^{-1}(0)$. As should be expected, the equality

$$\sum_{x \in V^{-1}(0)} \operatorname{sgn} x = \chi(\mathcal{H}) \tag{3.4}$$

holds, where $\chi(\mathcal{H})$ is the Euler characteristic of $\mathcal{H}$.

To prove equation (3.4), one has to understand continuous sections of the determinant line bundle. As we point out in Corollary A.6, the determinant line bundle of a family of Fredholm maps is canonically isomorphic to that of any Fredholm extension defined on the direct sum of the original bundle with some *oriented* finite-dimensional bundle $\mathcal{F}$. In the case at hand, we choose $\mathcal{F} = T\mathcal{H} \oplus T\mathcal{H}$ (which is canonically oriented even when $\mathcal{H}$ is not orientable) and extend the map

$$\mathcal{D} = \nabla^{\mathcal{H}}\sigma \colon T\mathcal{H} \longrightarrow T\mathcal{H}$$

to a surjective Fredholm bundle map

$$\begin{aligned} \widetilde{\mathcal{D}} \colon T\mathcal{H} \oplus (T\mathcal{H} \oplus T\mathcal{H}) &\longrightarrow T\mathcal{H} \\ (v, v', v'') &\longmapsto \mathcal{D}v - v' \,. \end{aligned}$$

Note that this is just a version of Example A.7 depending on the additional parameter $x \in \mathcal{H}$. In particular, we may consider the section

$$\mu \colon \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{D}et(\nabla^{\mathcal{H}}\sigma)$$

constructed by extension of the above μ_0 as in that example. Then, according to equation (A.8) of Example A.7, for any vector field V transverse to the zero section and any zero $x \in \mathcal{H}$ of V we have

$$\mu_{x,V} = \det((\nabla^{\mathcal{H}}\sigma)_{x,V})1 \otimes 1 = \det((\nabla V)_x)1 \otimes 1\,.$$

As $\operatorname{sgn} x$ is defined to be the sign of the coefficient of $1 \otimes 1$ in this formula, we see that it agrees with the index of the vector field V at the isolated zero $x \in \mathcal{H}$. Now (3.4) follows from the Poincaré-Hopf Theorem. In the context of this example, Corollary 3.5 expresses the familiar fact that on a manifold with non-zero Euler characteristic every vector field has at least one zero.

4. Counting formula

In this section, we continue to develop our general framework in the particular case of a Fredholm section of index 0. Then, under the assumptions of Theorem 3.4, the inverse image of a regular value of the projection map $\pi\colon s^{-1}(0) \longrightarrow \mathcal{B}$ is an oriented zero-dimensional manifold, whose oriented cobordism class D was proven to be independent of the regular value. Recall that integration of the constant function 1 identifies the set of oriented cobordism classes of oriented 0-dimensional manifolds with $\mathbf{Z}$, so we will think of D as an integer.

To state our result we need to introduce the notion of a section of gradient type. As in the previous section, we start with the Hilbert space bundle $\mathcal{E} \longrightarrow \mathcal{H} \times \mathcal{B}$ over the product of the Hilbert manifold $\mathcal{H}$ with the Banach manifold $\mathcal{B}$. We now further assume that we are given another Hilbert bundle $\mathcal{E}' \longrightarrow \mathcal{H}$ so that for any $x \in \mathcal{H}$ we have the scale of Hilbert spaces

$$T_x\mathcal{H} \hookrightarrow \mathcal{E}_{x,b} \hookrightarrow \mathcal{E}'_x \tag{4.1}$$

where $b \in \mathcal{B}$ is some given point. Furthermore assume $T_x\mathcal{H}$ is dense in $\mathcal{E}'_x$ for any $x \in \mathcal{H}$. Extending the inner product $\langle\cdot,\cdot\rangle'$ in $\mathcal{E}'_x$ to a dual pairing $\langle\cdot,\cdot\rangle$ between $T_x\mathcal{H}$, $(T_x\mathcal{H})^*$ and $\mathcal{E}_{x,b}$, $(\mathcal{E}_{x,b})^*$ one obtains the scale of Hilbert spaces

$$\mathcal{E}'_x \cong (\mathcal{E}'_x)^* \hookrightarrow (\mathcal{E}_{x,b})^* \hookrightarrow (T_x\mathcal{H})^*\,. \tag{4.2}$$

In particular, $T_x\mathcal{H} \hookrightarrow (\mathcal{E}_{x,b})^*$ densely.

We say that the section $s(\cdot,b)\colon \mathcal{H} \longrightarrow \mathcal{E}_{|\mathcal{H}\times\{b\}}$ is of gradient type with respect to the Hilbert scales (4.1) if there exists a C^4 functional $\mathcal{L}\colon \mathcal{H} \longrightarrow \mathbf{R}$ such that for any $x \in \mathcal{H}$, the differential $d_x\mathcal{L}\colon T_x\mathcal{H} \longrightarrow \mathbf{R}$ extends to a bounded linear functional $\operatorname{grad}_x \mathcal{L}\colon (\mathcal{E}_{x,b})^* \longrightarrow \mathbf{R}$, so that

$$\operatorname{grad}_x \mathcal{L} = s(x,b) \quad \forall x \in \mathcal{H}\,.$$

Moreover, we say that the critical set

$$C = Crit(\mathcal{L}) := \{x \in \mathcal{H} \mid d_x\mathcal{L} = 0\}$$

of $\mathcal{L}$ is nondegenerate if C is a closed (so in particular finite dimensional) C^2 – manifold so that, for any $x \in C$, zero is isolated in the spectrum of the Hessian $\nabla d_x\mathcal{L}\colon T_x\mathcal{H} \longrightarrow \mathcal{E}_{x,b}$ and we have

$$Ker\, \nabla d_x\mathcal{L} = T_xC = Coker\, \nabla d_x\mathcal{L}\,. \tag{4.3}$$

This definition is modelled on the corresponding nondegeneracy condition for the energy functional in § 2. We remark that if a section $s(\cdot, b)\colon \mathcal{H} \longrightarrow \mathcal{E}_{|\mathcal{H}\times\{b\}}$ is of gradient type for some functional $\mathcal{L}$ with nondegenerate critical set, then

$$\mathcal{D}et(\nabla^{\mathcal{H}} s)_{|C\times\{b\}} = \Lambda^{\max} TC \otimes \Lambda^{\max} TC\,,$$

which is canonically trivial. If the section $s\colon \mathcal{H}\times\mathcal{B} \longrightarrow \mathcal{E}$ satisfies condition $(\mathcal{O})$ of § 3, i.e. $\mathcal{D}et(\nabla^{\mathcal{H}} s) \longrightarrow \mathcal{H}\times\mathcal{B}$ is trivial, we will refer to the trivialization whose restriction to $C\times\{b\}$ coincides with the above canonical one as the preferred trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$.

Our goal in this section is to prove

THEOREM 4.1. *Suppose $\mathcal{H}$ is a Hilbert manifold and $\mathcal{B}$ is a pathconnected Banach manifold, both of class C^r $(r \geq 3)$ and without boundary, and suppose that $\mathcal{E} \longrightarrow \mathcal{H}\times\mathcal{B}$ is a C^r-Hilbert space bundle. Assume that $s\colon \mathcal{H}\times\mathcal{B} \longrightarrow \mathcal{E}$ is a C^l-section $(3 \leq l \leq r)$ satisfying assumptions $(\mathcal{R}), (\mathcal{F}_0), (\mathcal{O})$ and $(\mathcal{P})$ of § 3. Further assume that for some $b \in \mathcal{B}$, the restriction of s to $\mathcal{H}\times\{b\}$ is of gradient type with respect to the scale of Hilbert bundles $T\mathcal{H} \hookrightarrow \mathcal{E}_{\cdot,b} \hookrightarrow \mathcal{E}'$, i.e. $s(x,b) = \operatorname{grad}_x \mathcal{L}$ where $\mathcal{L}\colon \mathcal{H} \longrightarrow \mathbf{R}$ is a C^{l+1}-functional whose critical set $Crit(\mathcal{L})$ consists of a nondegenerate minimum.*

Then the oriented cobordism class $D \in \mathbf{Z}$ of the inverse image of a regular value of $\pi\colon s^{-1}(0) \longrightarrow \mathcal{B}$ with respect to the preferred trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}} s)$ is given by the Euler characteristic of $Crit(\mathcal{L})$, i.e.

$$D = \chi(Crit(\mathcal{L}))\,. \tag{4.4}$$

The proof of Theorem 4.1 is based on ideas from [CMS], where a vastly more general statement is proven. As the situation at hand is much more elementary, we prefer to give a shorter, more direct argument. As a first step we show

LEMMA 4.2. *Under the assumptions of Theorem 4.1, there exist a neighborhood $\mathcal{U}$ of $Crit(\mathcal{L})$ and a C^2-family of perturbations $\mathcal{L}_t\colon \mathcal{H} \longrightarrow \mathbf{R}$ of $\mathcal{L}$, $0 \leq t \leq 1$, with gradient $\operatorname{grad}_x\mathcal{L}_t\colon T_x\mathcal{H} \longrightarrow \mathcal{E}_{x,b}$ such that*

1. *$\mathcal{L}_t = \mathcal{L}$ outside $\mathcal{U}$ for all $t \in [0,1]$;*
2. *for $t \leq 1/2$, we have $\mathcal{L}_t = \mathcal{L}$;*
3. *$Crit(\mathcal{L}_t)$ is compact for all $t \in [0,1]$, and $Crit(\mathcal{L}_1)$ consists of finitely many Morse critical points of finite index;*
4. *for any $x \in Crit(\mathcal{L})$ and $t \in [0,1]$, $\nabla \mathrm{grad}\, \mathcal{L}$ and $\nabla \mathrm{grad}\, \mathcal{L}_t$ agree on $T_x Crit(\mathcal{L})^{\perp} \subset T_x\mathcal{H}$, where the orthogonal complement is taken with respect to the inner product on $T_x\mathcal{H}$.*

Proof. Denote by $\pi\colon \nu \longrightarrow Crit(\mathcal{L})$ the normal bundle of $Crit(\mathcal{L}) \subset \mathcal{H}$, whose fiber at $x \in Crit(\mathcal{L})$ is given by $T_x Crit(\mathcal{L})^{\perp} \subset T_x\mathcal{H}$. Following Meyer's argument in [Me] for the C^∞-case, under our conditions one finds an open neighborhood $\mathcal{U} \supset Crit(\mathcal{L})$ and a C^2 parametrization $\varphi\colon \nu(\delta) \longrightarrow \mathcal{U}$ of $\mathcal{U}$ by a δ-neighborhood $\nu(\delta)$ of the zero section in ν, such that in these coordinates on $\mathcal{U}$ the functional $\mathcal{L}$ has the form

$$\mathcal{L}(\varphi(v)) = \mathcal{L}(Crit(\mathcal{L})) + \|v\|^2 .$$

Then, as in the finite-dimensional situation, one may pick a smooth Morse function $f\colon Crit(\mathcal{L}) \longrightarrow \mathbf{R}$ and consider the functional

$$\lambda_\varepsilon(v) = \mathcal{L}(v) + \varepsilon\rho(\|v\|) \cdot f(\pi(v)) ,$$

where $\rho\colon \mathbf{R} \longrightarrow [0,1]$ is a bump function which is equal to 1 for $t \leq \delta/4$ and equal to zero for $t \geq \delta/2$, so that λ_ε agrees with $\mathcal{L}$ outside $\mathcal{U}$. Note also that, when restricted to the $\delta/4$-ball in the fiber of ν over some point $x \in Crit(\mathcal{L})$, $\mathcal{L}$ and λ_ε differ only by a constant.

For all $\varepsilon > 0$ sufficiently small, the functional λ_ε has a finite number of Morse critical points, located on $Crit(\mathcal{L})$ and corresponding to the critical points of f. The index of such a critical point equals its index as a critical point for f. The family $\mathcal{L}_t$ is constructed by reparametrizing the line between $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{L}_1 = \lambda_\varepsilon$ for a fixed, sufficiently small $\varepsilon > 0$. □

Now we fix a regular value $y \in \mathcal{B}$ of π and a C^2 path $\gamma\colon [-1,0] \longrightarrow \mathcal{B}$ with $\gamma(-1) = y$ and $\gamma(t) = b$ for $-1/2 \leq t \leq 0$. We define the C^2 family $\{\mathcal{S}_t\}_{t\in[-1,1]}$ of Fredholm sections $\mathcal{S}_t\colon \mathcal{H} \longrightarrow \mathcal{E}$ by

$$\mathcal{S}_t(x) := \begin{cases} s(x,\gamma(t)) & \text{for } -1 \leq t \leq 0 \\ \mathrm{grad}\, \mathcal{L}_t & \text{for } 0 \leq t \leq 1 , \end{cases} \tag{4.5}$$

where $\mathcal{L}_t$ is the family of functionals constructed in Lemma 4.2. We will denote $\mathcal{S}_{\pm 1}$ by $\mathcal{S}_\pm$. By the property $(\mathcal{P})$ for the section s (see the assumption in Theorem 4.1), $\mathcal{S}_t^{-1}(0)$ is compact for any $-1 \leq t \leq 0$ whereas for

$0 \leq t \leq 1$, $\mathcal{S}_t^{-1}(0)$ is compact by part (3) of Lemma 4.2. As y is a regular value of π, $\mathcal{S}_-^{-1}(0)$ is a compact, oriented 0-dimensional manifold. The same is true by construction for $\mathcal{S}_+^{-1}(0)$. As $\mathcal{S}_t^{-1}(0) \subset \mathcal{S}_0^{-1}(0)$ for all $t > 0$, the preferred trivialization of the determinant line bundle of $\nabla^{\mathcal{H}} s$ over $\mathcal{H} \times \mathcal{B}$ induces a trivialization of $\mathcal{D}et(\nabla \mathcal{S}_t)$ over $\mathcal{S}_t^{-1}(0)$ for any $-1 \leq t \leq 1$ which in turn induces together with the standard trivialization of the tangent space $T[-1,1]$ a trivialization of $\mathcal{D}et(\nabla \mathcal{S})$, again referred to as the preferred trivialization. In particular, $\mathcal{S}_+^{-1}(0)$ inherits an orientation from this trivialization of $\mathcal{D}et(\nabla \mathcal{S}_t)$.

LEMMA 4.3. *In the above situation, the oriented cobordism class D' of $\mathcal{S}_+^{-1}(0)$ is given by the Euler characteristic of $Crit(\mathcal{L})$,*

$$D' = \chi(Crit(\mathcal{L}))\,.$$

Proof. As for any $0 \leq t \leq 1$, $\mathcal{S}_t^{-1}(0) \subset \mathcal{S}_0^{-1}(0) = Crit(\mathcal{L}) =: C$, it is enough to restrict our attention to what happens on this set in order to understand the orientation of $\mathcal{S}_+^{-1}(0)$. Recall that $\mathcal{L}\colon \mathcal{H} \longrightarrow \mathbf{R}$ is assumed to have a nondegenerate minimum, which implies that

$$TC = \mathrm{Ker}\,(\nabla \mathcal{S}_0)|_C \cong \mathrm{Coker}\,(\nabla \mathcal{S}_0)|_C\,.$$

Part (4) of Lemma 4.2 asserts that for $x \in C$ and $0 \leq t \leq 1$, $(\nabla \mathcal{S}_t)_x$ differs from $(\nabla \mathcal{S}_0)_x$ only on T_xC, where it is given by a suitable multiple of the covariant derivative of the gradient of the Morse function $f\colon C \longrightarrow \mathbf{R}$ used in the proof of Lemma 4.2. Thus we are back in a finite dimensional situation and we may argue as in Example 3.6 to see that

$$D' = \sum_{p \in Crit(f)} \mathrm{sgn}(\det(\nabla \text{ grad } f)) = \sum_{p \in Crit(f)} \mathrm{ind}_p(f) = \chi(C)\,. \qquad \square$$

Note that, alternatively, the family $\mathcal{S}_t\colon \mathcal{H} \longrightarrow \mathcal{E}$ of Fredholm sections of index 0 may be viewed as a Fredholm section $\mathcal{S}\colon \mathcal{H} \times [-1,1] \longrightarrow \mathcal{E}$ of index 1. As $\nabla \mathcal{S}$ is an extension of $\nabla \mathcal{S}_t$ in the sense of Corollary A.6, we see that the canonical trivialization of the tangent bundle $T[-1,1]$ of the interval $[-1,1]$ gives rise to an isomorphism $\mathcal{D}et(\nabla \mathcal{S}_t) \cong \mathcal{D}et(\nabla \mathcal{S})$.

In view of Lemma 4.3, to show Theorem 4.1 it remains to prove that the oriented manifolds $\mathcal{S}_-^{-1}(0)$ and $\mathcal{S}_+^{-1}(0)$ are orientedly cobordant. To this end, we first construct, following [CMS], a so called finite-dimensional reduction of the problem. A standard finite-dimensional transversality argument will then yield the required cobordism.

LEMMA 4.4. *Let $\mathcal{T}: \mathcal{H} \times [-1,1] \longrightarrow \mathcal{E}$ be a Fredholm section such that $\mathcal{T}^{-1}(0) \subset \mathcal{H} \times [-1,1]$ is compact. Then there exist a positive integer $N \in \mathbf{N}$ and a bundle map*

$$\Gamma: \underline{\mathbf{R}}^N \longrightarrow \mathcal{E},$$

where $\underline{\mathbf{R}}^N \longrightarrow \mathcal{H}$ is the trivial bundle, such that

$$\mathcal{E}_{x,t} = \operatorname{Im}(\nabla \mathcal{T})_{x,t} + \operatorname{Im}\Gamma_{x,t} \tag{4.6}$$

whenever $(x,t) \in \mathcal{T}^{-1}(0)$.

Proof. Let $(x_0,t_0) \in \mathcal{T}^{-1}(0)$ be given. As $(\nabla\mathcal{T})_{x_0,t_0}$ is Fredholm, its cokernel is finite-dimensional. Hence there exist some positive integer $L = L(x_0,t_0)$ and a map $\Theta: \mathbf{R}^L \longrightarrow \mathcal{E}_{x_0,t_0}$ such that

$$\mathcal{E}_{x_0,t_0} = \operatorname{Im}(\nabla\mathcal{T})_{x_0,t_0} + \operatorname{Im}\Theta .$$

Given a small neighborhood $\mathcal{U} \subset \mathcal{H} \times [-1,1]$ of (x_0,t_0), we may use a trivialization of $\mathcal{E}_{|\mathcal{U}}$ to extend Θ to a family of maps $\Theta_{x,t}: \mathbf{R}^L \longrightarrow \mathcal{E}_{x,t}$ with $(x,t) \in \mathcal{U}$. As the set of surjective Fredholm operators is open, we may shrink $\mathcal{U}$ in such a way that

$$\mathcal{E}_{x,t} = \operatorname{Im}(\nabla\mathcal{T})_{x,t} + \operatorname{Im}\Theta_{x,t}$$

for all $(x,t) \in \mathcal{U}$. By compactness, we find finitely many of these open sets $\{\mathcal{U}_1,\dots,\mathcal{U}_r\}$ that cover $\mathcal{T}^{-1}(0)$. Set $\mathbf{R}^N = \mathbf{R}^{L_1} \oplus \cdots \oplus \mathbf{R}^{L_r}$. Using a partition of unity $\{\rho_i\}$ we may construct the required map $\Gamma: \underline{\mathbf{R}}^N \longrightarrow \mathcal{E}$ as

$$\Gamma_{x,t}(v) = \sum_{i=1}^{r} \rho_i(x,t)\Theta^i_{x,t}(pr_i(v)),$$

where $pr_i: \mathbf{R}^N \longrightarrow \mathbf{R}^{L_i}$ is the obvious projection. □

Applying Lemma 4.4 to our Fredholm section $\mathcal{S}: \mathcal{H} \times [-1,1] \longrightarrow \mathcal{E}$ defined in (4.5), we obtain a finite dimensional oriented Hilbert space $\mathbf{R}^N$ and the family of maps $\Gamma_{x,t}$. Following [CMS, Prop. 7.7], for $\delta > 0$ and any neighborhood $\mathcal{U} \subset \mathcal{H} \times [-1,1]$ of $\mathcal{S}^{-1}(0)$ we consider the set

$$H = H(\mathcal{U},\delta) := \{(x,t,v) \in \mathcal{U} \times \mathbf{R}^N \mid \mathcal{S}(x,t) = \Gamma_{x,t}(v), \|v\| < \delta\},$$

and the map $\sigma: H \longrightarrow \mathbf{R}^N$ given by $\sigma(x,t,v) = v$.

LEMMA 4.5. *With the notation just described, for a sufficiently small neighbourhood $\mathcal{U}$ of $\mathcal{S}^{-1}(0)$ and a sufficiently small $\delta > 0$,*

1. *H is an $(N+1)$-dimensional manifold with boundary $\partial H = \partial_- H \sqcup \partial_+ H$, where $\partial_\pm H$ is contained in $\mathcal{H} \times \{\pm 1\} \times \mathbf{R}^N$, respectively,*
2. *the preferred trivialization of $\mathcal{D}et(\nabla\mathcal{S})$ - see the definition before Lemma 4.3 - gives rise to an orientation of H, and*
3. *$\mathcal{S}^{-1}(0) \times \{0\} = \sigma^{-1}(0)$.*
4. *$\sigma|_{\partial H}$ is transverse to zero.*
5. *The 0-dimensional compact manifold $\sigma|_{\partial H}^{-1}(0)$, with the orientation induced from those of ∂H and $\mathbf{R}^N$, is orientedly diffeomorphic to $\mathcal{S}_+^{-1}(0) \sqcup -\mathcal{S}_-^{-1}(0)$.*

Proof. To prove (1), note that H is the zero set of the section $\mathcal{S} - \Gamma$ of the bundle $\mathcal{E} \longrightarrow \mathcal{H} \times [-1, 1] \times \mathbf{R}^N$. As on the boundary of $\mathcal{H} \times [-1, 1]$ the section $\mathcal{S}$ itself is transverse to zero, we can choose $\mathcal{U}$ and δ small enough so that the restriction of $\mathcal{S} - \Gamma$ to $(\mathcal{U} \times B(0, \delta)) \cap (\mathcal{H} \times \{\pm 1\}) \times \mathbf{R}^N$ is also transverse to zero, which proves (1) for boundary points of H. Furthermore, Γ was constructed to make $\mathcal{S} - \Gamma$ transverse to the zero section on $\mathcal{S}^{-1}(0) \times \{0\}$, so by the compactness of $\mathcal{S}^{-1}(0)$, for sufficiently small $\mathcal{U}$ and δ, the section $\mathcal{S} - \Gamma$ is transverse to the zero section for all $(x, t, v) \in \mathcal{U} \times B(0, \delta)$. It follows that, with this choice of $\mathcal{U}$ and δ, H is a manifold of dimension equal to the Fredhom index of $\mathcal{S} - \Gamma$, which is $N + 1$, and (1) is proven.

As $\nabla(\mathcal{S} - \Gamma)_{x,t,v}$ is an extension of $\nabla\mathcal{S}_{x,t}$, we see from Corollary A.6 that $\mathcal{D}et(\nabla(\mathcal{S} - \Gamma))|_H \cong \mathcal{D}et(\nabla\mathcal{S})|_H$. For points in $(\mathcal{S} - \Gamma)^{-1}(0)$ the cokernel of $\nabla(\mathcal{S} - \Gamma)$ is trivial and the kernel equals the tangent space of H, so that assertion (2) follows. Part (3) is a direct consequence of the definitions.

To prove (4), it is enough to show injectivity of $d\sigma|_{\partial H}$, because $\dim \partial H = N$. Note first that for $p = (x, \pm 1, 0) \in \partial H$ we have

$$T_p \partial H = \{(\varphi, w) \in T_u \mathcal{H} \times \mathbf{R}^N \mid (\nabla\mathcal{S}_\pm)_x(\varphi) = \Gamma_{x,\pm 1}(w)\}\,.$$

As $d(\sigma|_{\partial H})_p(\varphi, w) = w$, we see that

$$\operatorname{Ker} d(\sigma|_{\partial H})_p = \{(\varphi, 0) \in T_p \partial H\} = \operatorname{Ker}(\nabla\mathcal{S}_\pm)_x \times \{0\}\,. \tag{4.7}$$

But as $\mathcal{S}_\pm$ are Fredholm sections of index 0 which are transverse to zero, $\operatorname{Ker}(\nabla\mathcal{S}_\pm)_x = 0$, hence $\operatorname{Ker} d(\sigma|_{\partial H})_p$ is trivial for all $p \in \sigma^{-1}(0)$, and (4) is proven.

Now (5) follows from (2), (3) and (4) and the use of the standard trivialization of the tangent bundle $T[-1, 1]$ in the isomorphism of $\mathcal{D}et(\nabla\mathcal{S}_t)$ and $\mathcal{D}et(\nabla\mathcal{S})$. □

The last statement of Lemma 4.5 shows, together with the preceeding discussion, that in order to prove Theorem 4.1 it suffices to show that $\sigma|_{\partial H}^{-1}(0)$ is orientedly nullcobordant. This is a consequence of the following standard finite-dimensional transversality statement applied to σ :

LEMMA 4.6. *Let K be a possibly noncompact $(N+r)$-dimensional oriented manifold with boundary and let $\tau \colon K \longrightarrow \mathbf{R}^N$ be a differentiable map. Suppose that $\tau|_{\partial K}$ is transverse to 0, and that $\tau^{-1}(0)$ is compact. Then there is a perturbation $\tau' \colon K \longrightarrow \mathbf{R}^N$ of τ such that*

1. *$\tau' = \tau$ in a neighborhood of ∂K.*
2. *${\tau'}^{-1}(0)$ is compact.*
3. *τ' is transverse to 0.*

It follows that ${\tau'}^{-1}(0)$ is an r-dimensional compact oriented submanifold of K with $\partial({\tau'}^{-1}(0)) = {\tau'}^{-1}_{|\partial K}(0) = \tau|_{\partial K}^{-1}(0)$. In particular, $\tau|_{\partial K}^{-1}(0)$ is orientedly nullcobordant.

Sketch of Proof. As $\tau^{-1}(0)$ is compact and $\tau|_{\partial K}$ is transverse to 0, the set of points in K where τ is not transverse to 0 admits an open neighborhood whose closure is compact and does not intersect ∂K. It is sufficient to perturb τ on this neighborhood. The map τ' is then obtained by standard arguments. □

5. APPLICATION TO PERTURBATIONS OF THE HARMONIC MAP EQUATION

We want to apply the abstract procedure of the previous two sections to study the perturbed harmonic map equation. Returning to the setup of §2, we fix a homotopy class ζ of maps between M and M' and study the space $\mathcal{H}_\zeta^{(k+2)}$ of maps in this homotopy class of Sobolev class H^{k+2}, where $k > 2 + \frac{n+n'}{2}$. We let $\mathcal{E}^{(k)} \longrightarrow \mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$ denote the C^l-Hilbert space bundle ($2 \leq l < k - \frac{n+n'}{2}$) with fibre $\mathcal{E}_{u,F}^{(k)} \equiv H^k(u^*TM')$ at a point $(u,F) \in \mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$. Note that the bundle has a natural connection ∇ induced from the Levi-Civita connection of M'. The map

$$(u,F) \mapsto \Phi_F(u) \equiv \Phi(u,F) = \tau(u(x)) + F(x,u(x)) \tag{5.1}$$

defines a section of this bundle of class C^l, where $2 \leq l < k - \frac{n+n'}{2}$ as above. Note that the moduli space $\mathcal{M}_\zeta^{(k)}$ of solutions to the perturbed harmonic map equation in this homotopy class can be expressed as $\mathcal{M}_\zeta^{(k)} = \Phi^{-1}(0)$. The next three lemmas verify the basic assumptions made in the Fredholm argument of section 3 for our specific section Φ.

First we compute the restriction $(\nabla^{\mathcal{F}}\Phi)_{u,F}\colon T_F\mathcal{F}^{(k)} \longrightarrow \mathcal{E}^{(k)}_{u,F}$ of the covariant derivative $(\nabla\Phi)_{u,F}$, evaluated at $F_1 \in T_F\mathcal{F}^{(k)} \equiv \mathcal{F}^{(k)}$, as

$$\begin{aligned}(\nabla^{\mathcal{F}}\Phi)_{u,F}\cdot F_1\,(x) &= \lim_{\varepsilon\longrightarrow 0} \frac{\Phi(u,(F+\varepsilon F_1)(x,u(x))) - \Phi(u,F(x,u(x)))}{\varepsilon}\\ &= F_1(x,u(x))\,.\end{aligned}$$

Now it is easy to show

LEMMA 5.1. *Under the assumptions stated above, the restriction* $(\nabla^{\mathcal{F}}\Phi)_{u,F}$ *of the covariant derivative* $(\nabla\Phi)_{u,F}$ *to* $T_F\mathcal{F}^{(k)}$ *has a right inverse for all* $(u,F) \in \mathcal{H}^{(k+2)}_{\zeta} \times \mathcal{F}^{(k)}$. *In particular, the section* Φ *of* $\mathcal{E}^{(k)}$ *satisfies condition* $(\mathcal{R})$.

Proof. For fixed $(u,F) \in \mathcal{H}^{(k+2)}_{\zeta} \times \mathcal{F}^{(k)}$, we are supposed to construct a bounded linear map $R\colon H^k(u^*TM') \longrightarrow \mathcal{F}^{(k)}$ such that $(\nabla^{\mathcal{F}}\Phi)_{u,F} \circ R(\varphi) = \varphi$ for all $\varphi \in H^k(u^*TM')$. In light of the above computation this means that we need $R(\varphi)(x,u(x)) = \varphi(x)$ for all $\varphi \in H^k(u^*TM')$ and all $x \in M$.

As $k > \frac{n+n'}{2}$, u is continuous. Hence there are finite coverings $\{Q_j\}_{1\le j\le N}$ and $\{Q'_j\}_{1\le j\le N}$ of M and M' by open charts such that for all $1 \le j \le N$ the image $u(Q_j)$ is compactly contained in Q'_j. Here Q_j is allowed to be empty. We may further assume that the restriction of TM' to Q'_j has been trivialized by a smooth map $p_j\colon TM'_{|Q'_j} \longrightarrow Q'_j \times \mathbf{R}^{n'}$ for each $1 \le j \le N$. Denote by $\overline{p}_j$ the composition of p_j followed by the projection onto $\mathbf{R}^{n'}$. We choose a smooth partition of unity $\{\chi_j\}_{1\le j\le N}$ subordinate to $\{Q_j\}_{1\le j\le N}$ and smooth cut-off functions $\{\chi'_j\}_{1\le j\le N}$ on M' with

$$u(Q_j) \subset \{\chi'_j = 1\} \quad\text{and}\quad \operatorname{supp}\chi'_j \subset Q'_j\,.$$

Now define bounded linear maps $R_j\colon H^k(u^*TM') \longrightarrow \mathcal{F}^{(k)}$ by

$$R_j(\varphi)(x,y) := \begin{cases} \chi_j(x)\chi'_j(y)p_j^{-1}(y,\overline{p}_j(\varphi(x))) & \text{for } (x,y) \in Q_j \times Q'_j\\ 0 & \text{otherwise}\,.\end{cases}$$

Notice in particular that $R_j(\varphi)(x,u(x)) = \chi_j(x)\varphi(x)$. The required right inverse $R\colon H^k(u^*TM') \longrightarrow \mathcal{F}^{(k)}$ is then given by

$$R(\varphi)(x,y) := \sum_{j=1}^{N} R_j(\varphi)(x,y)\,. \qquad \square$$

LEMMA 5.2. *Under the assumptions stated at the beginning of this section, the restriction* $(\nabla^{\mathcal{H}}\Phi)_{u,F}$ *of the covariant derivative* $(\nabla\Phi)_{u,F}$ *to* $T_u\mathcal{H}_\zeta^{(k+2)}$ *is a continuous family of Fredholm operators of index 0 on* $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$. *In particular, the section* Φ *of* $\mathcal{E}^{(k)}$ *satisfies condition* $(\mathcal{F}_0)$.

Proof. Let $(u,F) \in \mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$ be given. Using the description (2.2) in Lemma 2.1 for the covariant derivative of τ, we compute the covariant derivative $\nabla^{\mathcal{H}}\Phi$ as

$$(\nabla^{\mathcal{H}}\Phi)_{u,F} \cdot \xi = J_u\xi + \nabla^{\mathcal{H}}F \cdot \xi\,, \tag{5.2}$$

where $\xi \in H^{k+2}(u^*TM')$ and $\nabla^{\mathcal{H}}F \cdot \xi$ denotes the covariant derivative of F along u in the direction ξ. Thus $(\nabla^{\mathcal{H}}\Phi)_{u,F}$ is a lower order perturbation of the selfadjoint elliptic operator J_u. It follows that $(\nabla^{\mathcal{H}}\Phi)_{u,F}$ is a Fredholm operator of index 0. □

LEMMA 5.3. *Under the assumptions stated at the beginning of this section, the determinant line bundle* $\mathcal{D}et(\nabla^{\mathcal{H}}\Phi)$ *over* $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$ *admits a canonical trivialization. In particular, the section* Φ *of* $\mathcal{E}^{(k)}$, *when restricted to* $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$, *satisfies condition* $(\mathcal{O})$.

Proof. From Lemma 5.2 and Theorem A.5 we know that the determinant line bundle $\mathcal{D}et(\nabla^{\mathcal{H}}\Phi)$ is well-defined on all of $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$. By [ES] the heat flow gives rise to a deformation retract of $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$ to $Crit(E|_\zeta) \times \{0\}$. At points $(u,0) \in Crit(E|_\zeta) \times \{0\}$, the operator $\nabla^{\mathcal{H}}\Phi$ equals the nonnegative self adjoint operator J_u (compare (5.2) above). Hence, for every $t > 0$, $J_u + t\mathbf{I}\colon H^2(u^*TM') \longrightarrow L^2(u^*TM')$ is a positive selfadjoint linear isomorphism and so in particular Fredholm. Thus the family $\{J_u\}_{u\in Crit(E|_\zeta)}$ of Fredholm operators is homotopic to the family $\{J_u + \mathbf{I}\}_{u\in Crit(E|_\zeta)}$ of bijective selfadjoint Fredholm operators, whose determinant line bundle is canonically trivial. □

REMARK 5.4. If the restriction of the energy functional to $\mathcal{H}_\zeta^{(k+2)}$ has a nondegenerate minimum, then the trivialization of Lemma 5.3 agrees with the preferred trivialization introduced in §4.

Following the outline of §3, we introduce the projection map

$$\pi \equiv \pi_{\zeta,k}\colon \mathcal{M}_\zeta^{(k)} = \Phi^{-1}(0) \longrightarrow \mathcal{F}^{(k)}\,.$$

We know from [KKS1] (cf. Theorem 1.2 (1) in the introduction) that π is proper. Combining this fact with Lemmas 5.1 through 5.3 and Theorem 3.4, we arrive at

THEOREM 5.5. *Let ζ be a homotopy class of maps between closed Riemannian manifolds M and M', where M' has nonpositive sectional curvature. Then for any $k > 2 + (n + n')/2$ the following statements hold:*

1. *The moduli space $\mathcal{M}_\zeta^{(k)}$ is a C^l-Hilbert submanifold of $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$ without boundary for any $0 \leq l < k - (n + n')/2$ and so the projection $\pi\colon \mathcal{M}_\zeta^{(k)} \longrightarrow \mathcal{F}^{(k)}$ is C^l.*
2. *The set $\pi^{-1}(F)$ of solutions of the perturbed harmonic map equation* (1.1) *is a compact 0-dimensional manifold, i.e. a finite set of points, for any regular value $F \in \mathcal{F}^{(k)}$ of π.*
3. *The restriction of the determinant line bundle $\mathcal{D}et(\nabla^{\mathcal{H}}\Phi)$ to $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)}$ is canonically trivial.*
4. *Assume in addition that some trivialization of the restriction of the determinant line bundle $\mathcal{D}et(\nabla^{\mathcal{H}}\Phi)$ to $\mathcal{M}_\zeta^{(k)}$ has been chosen. Then the preimage $\pi^{-1}(F)$ of any regular value F of π is oriented, and its oriented cobordism class D_ζ^k is independent of the choice of the regular value F.* □

Note that elliptic regularity implies that any regular value $F \in \mathcal{F}^{(k)}$ of $\pi_{\zeta,k}$ is also a regular value for $\pi_{\zeta,k'}$ for all $k' \leq k$. In particular, $D_\zeta = D_\zeta^k$ is independent of k. So together with Corollary 3.5 we have proven the first part of Theorem 1.1. Furthermore, assuming that $E|_\zeta$ has a nondegenerate minimum, the section Φ of $\mathcal{E}^{(k)}$ satisfies the assumptions of Theorem 4.1 and we obtain

THEOREM 5.6. *If the assumptions of Theorem 5.5 hold and the restriction $E|_\zeta$ of the energy functional has a nondegenerate minimum, then for the canonical trivialization of $\mathcal{D}et(\nabla^{\mathcal{H}}\Phi)$ we have*

$$D_\zeta = \chi(Crit(E|_\zeta)) . \qquad \square$$

We briefly comment on the results for the general case of equation (1.2) where $G \neq 0$. Here we consider the section

$$(u, F, G) \mapsto \Psi(u, F, G) = \tau(u(x)) + F(x, u(x)) + u_*(G(x, u(x)))$$

of the bundle $\mathcal{E}^{(k)} \longrightarrow \mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)} \times \mathcal{G}_c^{(k)}$ with fibre $\mathcal{E}_{u,F,G}^{(k)} = H^k(u^*TM')$. The right inverse R for $(\nabla^{\mathcal{F}}\Phi)_{u,F}$ constructed in Lemma 5.1, when considered as a map into $\mathcal{F}^{(k)} \times \{0\} \subset \mathcal{F}^k \times \mathcal{G}^k$, is obviously a right inverse for $(\nabla^{\mathcal{F}\times\mathcal{G}}\Psi)_{u,F,G}$. Similarly, as $\mathcal{G}^{(k)}$ is contractible, the same argument as in Lemma 5.3 shows that the determinant line bundle $\mathcal{D}et(\nabla^{\mathcal{H}}\Psi)$ is trivial over $\mathcal{H}_\zeta^{(k+2)} \times \mathcal{F}^{(k)} \times \mathcal{G}^{(k)}$.

Combining this with the obvious generalization of Lemma 5.2, we thus find that in the general case conditions $(\mathcal{R})$, $(\mathcal{F}_0)$ and $(\mathcal{O})$ hold as well.

Finally condition $(\mathcal{P})$ is true by Theorem 1.2 (2) (for c sufficiently small). Thus the version of Theorem 5.5, stated in the second part of Theorem 1.1 holds.

A. Determinant line bundles

In this appendix, we present a construction of the determinant line bundle associated to a family of Fredholm operators. Our exposition is partially based on the corresponding appendix in [Sal].

Let $A\colon E_1 \longrightarrow E_2$ be a Fredholm operator between Hilbert spaces. We can identify the dual of the cokernel $(\operatorname{Coker} A)^* = (E_2/\operatorname{Im} A)^*$ with the orthogonal complement $(\operatorname{Im} A)^\perp$ of the image using the inner product. The determinant $\mathcal{D}et(A)$ of the operator A is the 1-dimensional real vector space defined by

$$\begin{aligned}\mathcal{D}et(A) :&= \Lambda^{\max}\operatorname{Ker} A \otimes \Lambda^{\max}(\operatorname{Coker} A)^* \\ &\cong \Lambda^{\max}\operatorname{Ker} A \otimes \Lambda^{\max}(\operatorname{Im} A)^\perp,\end{aligned}$$

where for any finite dimensional vector space E we use the abbreviation $\Lambda^{\max}E$ for the top exterior product $\Lambda^{\dim E}E$. The first expression is the standard definition for Fredholm maps between Banach spaces, but in our Hilbert space context we will always work with the second interpretation.

Let $\mathbf{R}^N$ denote the N-dimensional Hilbert space with the standard Euclidean inner product and the standard orientation. Recall that an orientation of the finite dimensional vector space E is given by a choice of a linear isomorphism $\Lambda^{\max}E \cong \mathbf{R}$.

We say that a linear map $\widetilde{A}\colon E_1 \oplus \mathbf{R}^N \longrightarrow E_2$ is an extension of the linear map $A\colon E_1 \longrightarrow E_2$ if $\widetilde{A}_{|E_1\times\{0\}} = A$. As a first observation we have

Lemma A.1.

1. *Let $A\colon E_1 \longrightarrow E_2$ be a Fredholm map of index k. Then any extension $\widetilde{A}\colon E_1 \oplus \mathbf{R}^N \longrightarrow E_2$ of A is again Fredholm and of index $k+N$.*
2. *For A and $\widetilde{A}$ as in (1), $\mathcal{D}et(A)$ and $\mathcal{D}et(\widetilde{A})$ are canonically isomorphic.*

Proof. (1) As $\operatorname{Im} A \subset \operatorname{Im}\widetilde{A}$ and $\dim\operatorname{Ker}\widetilde{A} \le \dim\operatorname{Ker} A + N$ we see that $\widetilde{A}$ is Fredholm. The statement about the index is seen by writing $A = \widetilde{A}\circ\iota$

where $\iota\colon E_1 \longrightarrow E_1 \oplus \mathbf{R}^N$ is the standard embedding with Fredholm index $-N$ and using the additivity of the index under composition.

(2) As $\operatorname{Ker} A \subset \operatorname{Ker} \widetilde{A}$ and $(\operatorname{Im} \widetilde{A})^\perp \subset (\operatorname{Im} A)^\perp$, we have splittings

$$\text{(A.1)} \qquad \operatorname{Ker} \widetilde{A} \cong \operatorname{Ker} A \oplus \operatorname{Ker} \widetilde{A} \cap (\operatorname{Ker} A)^\perp$$

$$\text{(A.2)} \qquad (\operatorname{Im} A)^\perp \cong (\operatorname{Im} \widetilde{A})^\perp \oplus (\operatorname{Im} A)^\perp \cap \operatorname{Im} \widetilde{A}$$

where the orthogonal complement in (A.1) is taken in $E_1 \oplus \mathbf{R}^N$. Note that the orthogonal projection $P\colon E_1 \oplus \mathbf{R}^N \longrightarrow \mathbf{R}^N$ gives rise to an isomorphism

$$\text{(A.3)} \qquad \operatorname{Ker} \widetilde{A} \cap (\operatorname{Ker} A)^\perp \cong P(\operatorname{Ker} \widetilde{A}).$$

Denoting by $Q\colon \operatorname{Im} \widetilde{A} \longrightarrow (\operatorname{Im} A)^\perp \cap \operatorname{Im} \widetilde{A}$ the orthogonal projection, we claim that

$$\text{(A.4)} \qquad 0 \longrightarrow P(\operatorname{Ker} \widetilde{A}) \longrightarrow \mathbf{R}^N \xrightarrow{Q \circ \widetilde{A}|_{\mathbf{R}^N}} (\operatorname{Im} A)^\perp \cap \operatorname{Im} \widetilde{A} \longrightarrow 0$$

is a short exact sequence. Clearly, $Q \circ \widetilde{A}|_{\mathbf{R}^N}$ is onto. On the other hand, $Q(\widetilde{A}(0, v)) = 0$ is equivalent to $\widetilde{A}(0, v) = A(e)$ for some $e \in E_1$ and so $(-e, v) \in \operatorname{Ker} \widetilde{A}$. This in turn means that $v \in P(\operatorname{Ker} \widetilde{A})$, and so exactness also follows. Observing that for finite dimensional Hilbert spaces there are canonical isomorphisms

$$\Lambda^{\max} E \otimes \Lambda^{\max} F \cong \Lambda^{\max}(E \oplus F)$$

and

$$\Lambda^{\max} E \otimes \Lambda^{\max} E \cong \mathbf{R},$$

we combine equations (A.1) through (A.4) to obtain the canonical isomorphism

$$\begin{aligned} \mathcal{D}et(\widetilde{A}) &\cong \mathcal{D}et(\widetilde{A}) \otimes \Lambda^{\max} \mathbf{R}^N \\ &\cong \Lambda^{\max} \operatorname{Ker} \widetilde{A} \otimes \Lambda^{\max} (\operatorname{Im} \widetilde{A})^\perp \otimes \Lambda^{\max}\big((\operatorname{Im} A)^\perp \cap \operatorname{Im} \widetilde{A}\big) \otimes \Lambda^{\max} P(\operatorname{Ker} \widetilde{A}) \\ &\cong \Lambda^{\max} \operatorname{Ker} A \otimes \Lambda^{\max} P(\operatorname{Ker} \widetilde{A}) \otimes \Lambda^{\max} (\operatorname{Im} A)^\perp \otimes \Lambda^{\max} P(\operatorname{Ker} \widetilde{A}) \\ \text{(A.5)} \quad &\cong \mathcal{D}et(A). \qquad \square \end{aligned}$$

The following example serves to illustrate the isomorphism (A.5).

EXAMPLE A.2. Let $E_1 = E_2 = E$ be a finite dimensional Hilbert space and let $\mathbf{R}^N := E \oplus E$ be the canonically oriented sum of two copies of E. Given a linear map $T\colon E \longrightarrow E$, define

$$\begin{aligned} \widetilde{T}\colon E \oplus \mathbf{R}^N &\longrightarrow E \\ (w, (w', w'')) &\mapsto Tw - w'. \end{aligned}$$

Note that $\widetilde{T}$ is onto, so that $\mathcal{D}et(\widetilde{T}) \cong \Lambda^{\max}\operatorname{Ker}\widetilde{T}$. Given an orthonormal basis $\{e_1, \dots, e_k\}$ of $\operatorname{Ker} T$ and an orthonormal basis $\{f_1, \dots, f_k\}$ of $(\operatorname{Im} T)^\perp$, the element

$$\mu = e_1 \wedge \cdots \wedge e_k \otimes f_1 \wedge \cdots \wedge f_k \tag{A.6}$$

spans $\mathcal{D}et(T)$. Choosing any orthonormal basis $\{f_{k+1}, \dots, f_n\}$ of $\operatorname{Im} T$, there are unique elements $h_{k+1}, \dots, h_n \in (\operatorname{Ker} T)^\perp$ with

$$Th_l = f_l, \quad k+1 \leq l \leq n .$$

Then we claim that the isomorphism (A.5) is realized by mapping $\mu \in \mathcal{D}et(T)$ to

$$\widetilde{\mu} = \bigwedge_{i=1}^{k}(e_i, 0, 0) \wedge \bigwedge_{l=k+1}^{n} (h_l, f_l, 0) \wedge \bigwedge_{j=1}^{n}(0, 0, f_j) \in \mathcal{D}et(\widetilde{T}) . \tag{A.7}$$

First observe that the assignment $\mu \mapsto \widetilde{\mu}$ gives rise to a linear map which is independent of the choice of orthonormal bases $\{e_i\}_{1\leq i\leq k}$ and $\{f_j\}_{1\leq j\leq n}$. If $\{e_i'\}_{1\leq i\leq k}$ and $\{f_j'\}_{1\leq j\leq n}$ are two other orthonormal bases for $\operatorname{Ker} T$ and E respectively, such that $\{f_j'\}_{1\leq j\leq k}$ is a basis of $(\operatorname{Im} T)^\perp$, then there are orthogonal transformations $\alpha\colon \operatorname{Ker} T \longrightarrow \operatorname{Ker} T$ and $\beta : E \longrightarrow E$ with

$$\alpha(e_i) = e_i', \quad \beta(f_j) = f_j' .$$

Note that β preserves the splitting $E = \operatorname{Im} T \oplus (\operatorname{Im} T)^\perp$. We conclude that

$$\mu' = \det(\alpha) \det(\beta_{|(\operatorname{Im} T)^\perp})\, \mu$$

and

$$\widetilde{\mu}' = \det(\alpha) \det(\beta|_{\operatorname{Im} T}) \det(\beta)\, \widetilde{\mu} ,$$

so the claimed independence follows since $\det(\beta) = \det(\beta|_{\operatorname{Im} T}) \det(\beta_{|(\operatorname{Im} T)^\perp})$ and all three take values in $\{\pm 1\}$.

To compare our construction of the assigment $\mu \mapsto \widetilde{\mu}$ with the isomorphism (A.5), first note that in this example

$$P(\operatorname{Ker} \widetilde{T}) = \operatorname{Im} T \oplus E \subset E \oplus E \equiv \mathbf{R}^N .$$

Hence for a given μ as in (A.6), the choice of orthonormal basis $\{f_{k+1}, \dots, f_n\}$ of $\operatorname{Im} T$ corresponds to fixing the respresentative

$$\bigwedge_{l=k+1}^{n} (f_l, 0) \wedge \bigwedge_{j=1}^{n}(0, f_j) \otimes \bigwedge_{l=k+1}^{n} (f_l, 0) \wedge \bigwedge_{j=1}^{n}(0, f_j)$$

for the canonical generator of $\Lambda^{\max}P(\operatorname{Ker}\widetilde{T})\otimes\Lambda^{\max}P(\operatorname{Ker}\widetilde{T})$. The passage from $(f_l,0)$ to $(h_l,f_l,0)$ realizes the identification of $P(\operatorname{Ker}\widetilde{T})$ with $\operatorname{Ker}\widetilde{T}\cap(\operatorname{Ker}T)^{\perp}$ as in (A.3). Because $\widetilde{T}$ is surjective, we have $(\operatorname{Im}T)^{\perp}\cong(\operatorname{Im}T)^{\perp}\cap\operatorname{Im}\widetilde{T}$, and the identification

$$\Lambda^{\max}(\operatorname{Im}T)^{\perp}\otimes\Lambda^{\max}P(\operatorname{Ker}\widetilde{T})\cong\Lambda^{\max}\mathbf{R}^N$$

corresponding to the splitting (A.4), is realized by lifting the generator $f_1\wedge\cdots\wedge f_k$ of $\Lambda^{\max}(\operatorname{Im}T)^{\perp}$ to the generator $(f_1,0)\wedge\cdots\wedge(f_k,0)$ of $\Lambda^{\max}(P(\operatorname{Ker}\widetilde{T}))^{\perp}$. By wedging with $\bigwedge_{l=k+1}^{n}(f_l,0)\wedge\bigwedge_{j=1}^{n}(0,f_j)\in\Lambda^{\max}P(\operatorname{Ker}\widetilde{T})$ this gives a representative of the canonical generator of $\Lambda^{\max}\mathbf{R}^N$, and our discussion of Example A.2 is complete.

Now let $\mathcal{E}_1\longrightarrow X$ and $\mathcal{E}_2\longrightarrow X$ be continuous Hilbert space bundles over some Hausdorff topological space X, and let $\mathcal{A}\colon\mathcal{E}_1\longrightarrow\mathcal{E}_2$ be a Fredholm bundle map inducing the identity on X, i.e. a continuous family of Fredholm maps $\mathcal{A}_x\colon\mathcal{E}_{1,x}\longrightarrow\mathcal{E}_{2,x}$. If $\dim\operatorname{Ker}A_x$ is constant or, equivalently, $\dim(\operatorname{Im}A_x)^{\perp}$ is constant, we will say the bundle map has *constant rank.* This terminology is motivated by the finite-dimensional case. For constant rank Fredholm bundle maps, we denote by $\operatorname{Ker}\mathcal{A}$ and $(\operatorname{Im}\mathcal{A})^{\perp}$ the corresponding finite-dimensional vector bundles over X.

A constant rank extension of the Fredholm bundle map $\mathcal{A}\colon\mathcal{E}_1\longrightarrow\mathcal{E}_2$ is a constant rank bundle map $\widetilde{\mathcal{A}}\colon\mathcal{E}_1\oplus\mathcal{F}\longrightarrow\mathcal{E}_2$, defined on the direct sum of $\mathcal{E}_1$ with a finite-dimensional *oriented* vector bundle $\mathcal{F}$ over X, such that $\widetilde{\mathcal{A}}_{|\mathcal{E}_1}=\mathcal{A}$. Often one chooses $\mathcal{F}=\underline{\mathbf{R}}^N$, the trivial bundle over X with fiber $\mathbf{R}^N$ and with the standard orientation.

LEMMA A.3. *Let $\mathcal{A}\colon\mathcal{E}_1\longrightarrow\mathcal{E}_2$ be a Fredholm bundle map between Hilbert space bundles over some base space X, and let $x\in X$ be any point. Then there is a neighborhood $U_x\subset X$ of x and some $N\geq 0$ such that $\mathcal{A}|_{U_x}$ admits a constant rank extension $\widetilde{\mathcal{A}}\colon\mathcal{E}_{1|U_x}\oplus\underline{\mathbf{R}}^N\longrightarrow\mathcal{E}_{2|U_x}$.*

Proof. Let $V\subset X$ be a neighborhood of $x\in X$ such that both bundles are trivialized over V, i.e. $\mathcal{E}_{i|V}\cong V\times\mathcal{E}_{i,x}$ for $i\in\{1,2\}$. As $\mathcal{A}_x\colon\mathcal{E}_{1,x}\longrightarrow\mathcal{E}_{2,x}$ is Fredholm, we can choose a basis $w_1,\dots,w_N$ of $(\operatorname{Im}\mathcal{A}_x)^{\perp}$ and define a map $\widetilde{\mathcal{A}}\colon V\times\mathcal{E}_{1,x}\times\mathbf{R}^N\longrightarrow V\times\mathcal{E}_{2,x}$ by

$$\widetilde{\mathcal{A}}(y,v_y,\sum_{j=1}^{N}\alpha_je_j):=(y,\mathcal{A}_yv_y+\sum_{j=1}^{N}\alpha_jw_j)\,.$$

By construction, $\widetilde{\mathcal{A}}_x \equiv \widetilde{\mathcal{A}}(x,\cdot,\cdot)$ is surjective. As $\widetilde{\mathcal{A}}_y$ depends continuously on y and the set of surjective linear maps is open, there is a whole neighborhood $U_x \subset V$ of x such that $\widetilde{\mathcal{A}}_y$ is surjective for $y \in U_x$. The family $\widetilde{\mathcal{A}}|_{U_x}$ of Fredholm operators has constant index, and hence, since it is surjective, it has constant rank. □

LEMMA A.4. *Let $\widetilde{\mathcal{A}}_1 \colon \mathcal{E}_1 \oplus \underline{\mathbf{R}}^{N_1} \longrightarrow \mathcal{E}_2$ and $\widetilde{\mathcal{A}}_2 \colon \mathcal{E}_1 \oplus \underline{\mathbf{R}}^{N_2} \longrightarrow \mathcal{E}_2$ be constant rank extensions of $\mathcal{A} \colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ defined on subsets U_1 and U_2 of X, respectively. Then on the intersection $U_1 \cap U_2$ we have a canonical bundle isomorphism*

$$\Lambda^{\max}\operatorname{Ker}\widetilde{\mathcal{A}}_1 \otimes \Lambda^{\max}(\operatorname{Im}\widetilde{\mathcal{A}}_1)^{\perp} \cong \Lambda^{\max}\operatorname{Ker}\widetilde{\mathcal{A}}_2 \otimes \Lambda^{\max}(\operatorname{Im}\widetilde{\mathcal{A}}_2)^{\perp}\,.$$

Proof. Denote by $\widetilde{\mathcal{A}}$ the common extension $\widetilde{\mathcal{A}} \colon \mathcal{E}_1 \oplus \underline{\mathbf{R}}^{N_1} \oplus \underline{\mathbf{R}}^{N_2} \longrightarrow \mathcal{E}_2$ of $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ on $U_1 \cap U_2$, which in general needs not be of constant rank. However, given any point $x \in U_1 \cap U_2$, by Lemma A.3 there exists a constant rank extension $\widehat{\mathcal{A}}$ of $\widetilde{\mathcal{A}}$ in a neighborhood U_x of x. As $\widehat{\mathcal{A}}$ extends both $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$, we can use part (2) of Lemma A.1 to conclude that, pointwise and hence everywhere on U_x, we have canonical isomorphisms

$$\begin{aligned}\Lambda^{\max}\operatorname{Ker}\widetilde{\mathcal{A}}_1 \otimes \Lambda^{\max}(\operatorname{Im}\widetilde{\mathcal{A}}_1)^{\perp} &\cong \Lambda^{\max}\operatorname{Ker}\widehat{\mathcal{A}} \otimes \Lambda^{\max}(\operatorname{Im}\widehat{\mathcal{A}})^{\perp}\\ &\cong \Lambda^{\max}\operatorname{Ker}\widetilde{\mathcal{A}}_2 \otimes \Lambda^{\max}(\operatorname{Im}\widetilde{\mathcal{A}}_2)^{\perp}.\end{aligned}$$

As $x \in U_1 \cap U_2$ was arbitrary, this proves the claim. □

We are now in a position to formulate

THEOREM A.5. *Let $\mathcal{A} \colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be a Fredholm bundle map between Hilbert space bundles over some base space X. Then there is a well-defined line bundle over X, called the* determinant line bundle $\mathcal{D}et(\mathcal{A})$, *whose restriction to any open subset $U \subset X$ where $\mathcal{A}$ has a constant rank extension $\widetilde{\mathcal{A}}$ is given as*

$$\mathcal{D}et(\mathcal{A})_{|U} \cong \Lambda^{\max}\operatorname{Ker}\widetilde{\mathcal{A}} \otimes \Lambda^{\max}(\operatorname{Im}\widetilde{\mathcal{A}})^{\perp}\,.$$

Proof. Consider the collection $\{\widetilde{\mathcal{A}}_i\}_{i\in I}$ of all local constant rank extensions $\widetilde{\mathcal{A}}_i \colon \mathcal{E}_1 \oplus \underline{\mathbf{R}}^{N_i} \longrightarrow \mathcal{E}_2$ of $\mathcal{A}$, each defined over some open subset $U_i \subset X$. By Lemma A.3, the collection of open subsets U_i covers all of X, and by Lemma A.4 the local determinant bundles are compatible on overlaps. Thus they define a global line bundle. □

The following fact is apparent from the definitions.

COROLLARY A.6. *Let* $\widetilde{\mathcal{A}}\colon \mathcal{E}_1 \oplus \mathcal{F} \longrightarrow \mathcal{E}_2$ *be an extension of the Fredholm bundle map* $\mathcal{A}\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$*, where* $\mathcal{F}$ *is a finite-dimensional oriented vector bundle over* X*. Then the orientation of* $\mathcal{F}$ *gives rise to a canonical isomorphism*

$$\mathcal{D}et(\mathcal{A}) \cong \mathcal{D}et(\widetilde{\mathcal{A}}).$$

Proof. The pointwise isomorphism of Lemma A.1 is canonical and thus gives rise to a global canonical isomorphism. □

EXAMPLE A.7. Given a finite dimensional Hilbert space E, we consider the trivial E-bundle $\mathcal{E} = L(E,E) \times E \longrightarrow L(E,E)$ over the space of linear automorphisms of E with the tautological bundle map $\mathcal{T}\colon \mathcal{E} \longrightarrow \mathcal{E}$, which on the fiber $\mathcal{E}_T \equiv E$ over $T \in L(E,E)$ is given by $\mathcal{T}_T(w) = T(w)$.

As $L(E,E)$ is contractible, we conclude that the line bundle $\mathcal{D}et(\mathcal{T})$ is trivial. For use in Example 3.6, we want to construct an explicit trivialization. To this end, we consider the canonically oriented bundle $\mathcal{F} = \mathcal{E} \oplus \mathcal{E} \longrightarrow L(E,E)$ and the extension

$$\widetilde{\mathcal{T}}\colon \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{E}$$

of $\mathcal{T}$, which acts on the fibre over T by $\widetilde{\mathcal{T}}_T(w,(w',w'')) = T(w) - w'$. Note that this is just the family of maps $\widetilde{T}$ considered individually in Example A.2. For $T = 0$ we have

$$\mathcal{D}et(\mathcal{T}_0) = \Lambda^{\max}\mathrm{Ker}\,\mathcal{T}_0 \otimes \Lambda^{\max}(\mathrm{Im}\,\mathcal{T}_0)^{\perp} \cong \Lambda^{\max}E \otimes \Lambda^{\max}E,$$

and so any orthonormal basis $\{e_1,\dots,e_n\}$ of E gives rise to the canonical generator $\mu_0 = \bigwedge_{i=1}^{n} e_i \otimes \bigwedge_{j=1}^{n} e_j \in \mathcal{D}et(\mathcal{T}_0)$. As explained in Example A.2, the element $\mu_0 \in \mathcal{D}et(\mathcal{T}_0)$ is mapped to the generator

$$\widetilde{\mu}_0 = \bigwedge_{i=1}^{n}(e_i,0,0) \wedge \bigwedge_{j=1}^{n}(0,0,e_j) \in \mathcal{D}et(\widetilde{\mathcal{T}}_0)$$

under the isomorphism (A.5) identifying $\mathcal{D}et(\mathcal{T})$ and $\mathcal{D}et(\widetilde{\mathcal{T}})$. Observe that $\widetilde{\mu}_0$ can be continuously extended to a trivialization of $\mathcal{D}et(\widetilde{\mathcal{T}})$ by setting

$$\widetilde{\mu}_T = \bigwedge_{i=1}^{n}(e_i,Te_i,0) \wedge \bigwedge_{j=1}^{n}(0,0,e_j) \in \mathcal{D}et(\widetilde{\mathcal{T}}_T).$$

To see which element $\mu_T \in \mathcal{D}et(\mathcal{T}_T)$ this corresponds to under the isomorphism (A.5), we observe that $\widetilde{\mu}_T$ is independent of the chosen orthonormal basis. Hence we may arrange things so that $\{e_1,\dots,e_k\}$ span

$\operatorname{Ker} T$. Then, using the orthonormal basis $\{f_1, \ldots, f_n\}$ of E and the elements $\{h_{k+1}, \ldots, h_n\} \in (\operatorname{Ker} T)^\perp$ as in Example A.2, we see that

$$\widetilde{\mu}_T = \bigwedge_{i=1}^{k}(e_i, 0, 0) \wedge \bigwedge_{l=k+1}^{n} (e_l, Te_l, 0) \wedge \bigwedge_{j=1}^{n}(0, 0, e_j)$$

$$= c \cdot \bigwedge_{i=1}^{k}(e_i, 0, 0) \wedge \bigwedge_{r=k+1}^{n} (h_r, f_r, 0) \wedge \bigwedge_{s=1}^{n}(0, 0, f_s),$$

where

$$c = \det(\langle Te_l, f_r\rangle_{l,r=k+1}^{n}) \cdot \det(\langle e_j, f_s\rangle_{j,s=1}^{n}).$$

In particular, $k = 0$ in the above formula whenever T is invertible, so that the expression then simplifies to $c = \det(T)$.

To recapitulate the above, we have shown that there is a natural trivialization of $\mathcal{D}et(\mathcal{T})$ determined by the canonical generator of $\mathcal{D}et(\mathcal{T}_0)$, which when evaluated at an invertible map $T \in L(E, E)$ corresponds to the element

(A.8) $$\mu_T = \det(T)\, 1 \otimes 1 \in \mathcal{D}et(\mathcal{T}_T).$$

This explains the name determinant line bundle for the bundle $\mathcal{D}et(\mathcal{T})$ and concludes our discussion of Example A.7.

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