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SMOOTH LYAPUNOV 1-FORMS

by M. FARBER^{*}), T. KAPPELER, J. LATSCHEV[†]) and E. ZEHNDER

ABSTRACT. We find conditions which guarantee that a given flow Φ on a closed smooth manifold M admits a smooth Lyapunov 1-form ω lying in a prescribed de Rham cohomology class $\xi \in H^1(M; \mathbf{R})$. These conditions are formulated in terms of Schwartzman's asymptotic cycles $\mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$ of the flow.

1. INTRODUCTION

C. Conley [1, 2] showed that any continuous flow $\Phi: X \times \mathbf{R} \rightarrow X$ on a compact metric space X “*decomposes*” into a chain recurrent flow and a gradient-like flow. More precisely, he proved the existence of a continuous function $L: X \rightarrow \mathbf{R}$ which (i) decreases along any orbit of the flow in the complement $X - R$ of the chain recurrent set $R \subset X$ of Φ and (ii) is constant on the connected components of R . Such a function L is called a *Lyapunov function* for Φ . This existence result plays a fundamental role in Conley's program of understanding general flows as collections of isolated invariant sets linked by heteroclinic orbits.

A more general notion of a *Lyapunov 1-form* was introduced in paper [5]. Lyapunov 1-forms, as compared to Lyapunov functions, allow one to go one step further and to analyze the flow within the chain-recurrent set R as well. Lyapunov 1-forms provide an important tool in applying methods of homotopy theory to dynamical systems. In the recent papers [4], [5] a generalization of the Lusternik–Schnirelman theory was constructed which applies to flows admitting Lyapunov 1-forms.

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The problem of existence of Lyapunov 1-forms was addressed in our recent preprint [6], where we worked in the category of compact metric spaces, continuous flows and continuous closed 1-forms. In the present paper we study the smooth version of the problem: we construct smooth Lyapunov 1-forms for smooth flows on smooth manifolds. We use Schwartzman's asymptotic cycles to formulate a necessary condition for the existence of Lyapunov 1-forms in a given cohomology class. We also show that under an additional assumption this condition is equivalent to the homological condition introduced in our previous paper [6].

2. DEFINITION

Let V be a smooth vector field on a smooth manifold M . Assume that V generates a continuous flow $\Phi: M \times \mathbf{R} \rightarrow M$ and $Y \subset M$ is a closed, flow-invariant subset.

DEFINITION 1. A smooth closed 1-form ω on M is called a *Lyapunov 1-form* for the pair (Φ, Y) if it has the following properties:

- (A1) The function $\iota_V(\omega) = \omega(V)$ is negative on $M - Y$;
- (A2) There exists a smooth function $f: U \rightarrow \mathbf{R}$ defined on an open neighborhood U of Y such that

$$\omega|_U = df \quad \text{and} \quad df|_Y = 0.$$

The above definition is a modification of the notion of a Lyapunov 1-form introduced in section 6 of [5]. The definition of [5] requires that Y consists of finitely many points and the vector field V is locally a gradient of ω with respect to a Riemannian metric.

Definition 1 can also be compared with the definition of a Lyapunov 1-form in the continuous setting which was introduced in [6]. Condition (A1) above is slightly stronger than condition (L1) of Definition 1 in [6]. Condition (A2) is similar to condition (L2) of Definition 1 from [6] although they are not equivalent.

There are several natural alternatives for condition (A2). One of them is: (A2') The 1-form ω , viewed as a map $\omega: M \rightarrow T^*(M)$, vanishes on Y .

It is clear that (A2) implies (A2'). We can show that the converse is true under some additional assumptions:

LEMMA 1. *If the de Rham cohomology class ξ of ω is integral, $\xi = [\omega] \in H^1(M; \mathbf{Z})$, then the conditions $(\Lambda 2')$ and $(\Lambda 2)$ are equivalent.*

Proof. Clearly we only need to show that $(\Lambda 2')$ implies $(\Lambda 2)$. Since ξ is integral there exists a smooth map $\phi: M \rightarrow S^1$ such that $\omega = \phi^*(d\theta)$, where $d\theta$ is the standard angular 1-form on the circle S^1 . Let $\alpha \in S^1$ be a regular value of ϕ . Assuming that $(\Lambda 2')$ holds it then follows that $U = M - \phi^{-1}(\alpha)$ is an open neighborhood of Y . Clearly $\omega|_U = df$ where $f: U \rightarrow \mathbf{R}$ is a smooth function which is related to ϕ by $\phi(x) = \exp(if(x))$ for any $x \in U$. Hence $(\Lambda 2)$ holds.

LEMMA 2. *The conditions $(\Lambda 2')$ and $(\Lambda 2)$ are equivalent if Y is an Euclidean Neighborhood Retract (ENR).*

Proof. Again, we only have to establish $(\Lambda 2') \Rightarrow (\Lambda 2)$. Since Y is an ENR it admits an open neighbourhood $U \subset M$ such that the inclusion $i_U: U \rightarrow M$ is homotopic to $i_Y \circ r$, where $i_Y: Y \rightarrow M$ is the inclusion and $r: U \rightarrow Y$ is a retraction (see [3], chapter 4, §8, Corollary 8.7). Pick a base point x_j in every path-connected component U_j of U and define a smooth function $f_j: U_j \rightarrow \mathbf{R}$ by

$$f_j(x) = \int_{x_j}^x \omega, \quad x \in U_j.$$

The latter integral is independent of the choice of the integration path in U_j connecting x_j with x . This claim is equivalent to the vanishing of the integral $\int_\gamma \omega$ for any closed loop γ lying in U . To show this we apply the retraction to see that γ is homotopic in M to the loop $\gamma_1 = r \circ \gamma$, which lies in Y ; thus we obtain $\int_\gamma \omega = \int_{\gamma_1} \omega = 0$ because of $(\Lambda 2')$. It is clear that the functions f_j together determine a smooth function $f: U \rightarrow \mathbf{R}$ with $df = \omega|_U$.

A class of interesting examples can be obtained as follows. Let ω be a smooth closed 1-form on a closed Riemannian manifold M . Consider the negative gradient vector field V of ω , i.e. $\langle V, X \rangle = -\omega(X)$ for any vector field X on M where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric. Denote by Φ the flow induced by the vector field V and by Y the set of zeros of ω . Then clearly conditions $(\Lambda 1)$ and $(\Lambda 2')$ are satisfied. If either the cohomology class of ω is integral or Y is an ENR then (by the two Lemmas above) ω is a Lyapunov 1-form for the pair (Φ, Y) .

Our main goal in this paper is to find topological conditions which guarantee that for a given vector field V on M there exists a Lyapunov 1-form ω lying in a prescribed cohomology class $\xi \in H^1(M; \mathbf{R})$.

3. ASYMPTOTIC CYCLES OF SCHWARTZMAN

Let M be a closed smooth manifold and let V be a smooth vector field. Let $\Phi: M \times \mathbf{R} \rightarrow M$ be the flow generated by V .

Consider a Borel measure μ on M which is invariant under Φ . According to S. Schwartzman [16], these data determine a real homology class

$$\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$$

called *the asymptotic cycle of the flow Φ corresponding to the measure μ* . The class \mathcal{A}_μ is defined as follows. For a de Rham cohomology class $\xi \in H^1(M; \mathbf{R})$ the evaluation $\langle \xi, \mathcal{A}_\mu \rangle \in \mathbf{R}$ is given by the integral

$$(3.1) \quad \langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega) d\mu,$$

where ω is a closed 1-form in the class ξ . Note that $\langle \xi, \mathcal{A}_\mu \rangle$ is well-defined, i.e. it depends only on the cohomology class ξ of ω , see [16], p. 277. Indeed, replacing ω by $\omega' = \omega + df$, where $f: M \rightarrow \mathbf{R}$ is a smooth function, the integral in (3.1) gets changed by the quantity

$$(3.2) \quad \int_M V(f) d\mu = \lim_{s \rightarrow 0} \frac{1}{s} \int_M \{f(x \cdot s) - f(x)\} d\mu(x).$$

Here $V(f)$ denotes the derivative of f in the direction of the vector field V and $x \cdot s$ stands for the flow $\Phi(x, s)$ of the vector field V . Since the measure μ is flow invariant, the integral on the RHS of (3.2) vanishes for any f . It is clear that the RHS of (3.1) is a linear function of $\xi \in H^1(M; \mathbf{R})$. Hence there exists a unique real homology class $\mathcal{A}_\mu \in H_1(M; \mathbf{R})$ which satisfies (3.1) for all $\xi \in H^1(M; \mathbf{R})$.

4. NECESSARY CONDITIONS

We consider the flow Φ as being fixed and we vary the invariant measure μ . As the class $\mathcal{A}_\mu \in H_1(M; \mathbf{R})$ depends linearly on μ , the set of asymptotic cycles \mathcal{A}_μ corresponding to all Φ -invariant positive measures μ forms a convex cone in the vector space $H_1(M; \mathbf{R})$.

PROPOSITION 1. *Assume that there exists a Lyapunov 1-form for (Φ, Y) lying in a cohomology class $\xi \in H^1(M; \mathbf{R})$. Then*

$$(4.1) \quad \langle \xi, \mathcal{A}_\mu \rangle \leq 0$$

for any Φ -invariant positive Borel measure μ on M ; equality in (4.1) takes place if and only if the complement of Y has measure zero. Further, the restriction of ξ to Y , viewed as a Čech cohomology class

$$\xi|_Y \in \check{H}^1(Y; \mathbf{R})$$

vanishes, $\xi|_Y = 0$.

Proof. Let ω be a Lyapunov 1-form for (Φ, Y) lying in the class ξ . According to Definition 1, the function $\iota_V(\omega)$ is negative on $M - Y$ and vanishes on Y . We obtain that the integral

$$\int_M \iota_V(\omega) d\mu = \langle \xi, \mathcal{A}_\mu \rangle$$

is nonpositive.

Assuming $\mu(M - Y) > 0$, we find a compact $K \subset M - Y$ with $\mu(K) > 0$; this follows from the Theorem of Riesz – see e.g. [12], Theorem 2.3(iv), p. 256. There is a constant $\epsilon > 0$ such that $\iota_V(\omega)|_K \leq -\epsilon$. Therefore, one has

$$\int_M \iota_V(\omega) d\mu \leq -\epsilon\mu(K) < 0.$$

Hence, the value $\langle \xi, \mathcal{A}_\mu \rangle$ is strictly negative if the measure μ is not supported in Y .

To prove the second statement we observe (see [19]) that the Čech cohomology $\check{H}^1(Y; \mathbf{R})$ equals the direct limit of the singular cohomology

$$\check{H}^1(Y; \mathbf{R}) = \lim_{W \supset Y} H^1(W; \mathbf{R}),$$

where W runs over open neighborhoods of Y . It is clear in view of condition $(\Lambda 2)$ that $\xi|_U = 0 \in H^1(U; \mathbf{R})$ (by the de Rham theorem). Hence the result follows.

5. CHAIN-RECURRENT SET R_ξ

Given a flow Φ , our aim is to construct a Lyapunov 1-form ω for a pair (Φ, Y) lying in a given cohomology class $\xi \in H^1(M; \mathbf{R})$. A natural candidate

for Y is the subset $R_\xi = R_\xi(\Phi)$ of the chain-recurrent set $R = R(\Phi)$ which was defined in [6]. For convenience of the reader we briefly recall the definition.

Fix a Riemannian metric on M and denote by d the corresponding distance function. Given any $\delta > 0$, $T > 1$, a (δ, T) -chain from $x \in M$ to $y \in M$ is a finite sequence $x_0 = x, x_1, \dots, x_N = y$ of points in M and numbers $t_1, \dots, t_N \in \mathbf{R}$ such that $t_i \geq T$ and $d(x_{i-1} \cdot t_i, x_i) < \delta$ for all $1 \leq i \leq N$. Here we use the notation $\Phi(x, t) = x \cdot t$. The chain recurrent set $R = R(\Phi)$ of the flow Φ is defined as the set of all points $x \in M$ such that for any $\delta > 0$ and $T > 1$ there exists a (δ, T) -chain starting and ending at x . The chain recurrent set is closed and invariant under the flow.

Given a cohomology class $\xi \in H^1(M; \mathbf{R})$ there is a natural covering space $p_\xi: \tilde{M}_\xi \rightarrow M$ associated with ξ . A closed loop $\gamma: [0, 1] \rightarrow M$ lifts to a closed loop in \tilde{M}_ξ if and only if the value of the cohomology class ξ on the homology class $[\gamma] \in H_1(M; \mathbf{Z})$ vanishes, $\langle \xi, [\gamma] \rangle = 0$. See [19].

The flow Φ lifts uniquely to a flow $\tilde{\Phi}$ on the covering \tilde{M}_ξ . Consider the chain recurrent set $R(\tilde{\Phi}) \subset \tilde{M}_\xi$ of the lifted flow and denote by $R_\xi = p_\xi(R(\tilde{\Phi})) \subset M$ its projection onto M . The set R_ξ is referred to as the chain recurrent set associated to the cohomology class ξ . It is clear that R_ξ is a closed and Φ -invariant subset of R . We denote by C_ξ the complement of R_ξ in R ,

$$C_\xi = R - R_\xi.$$

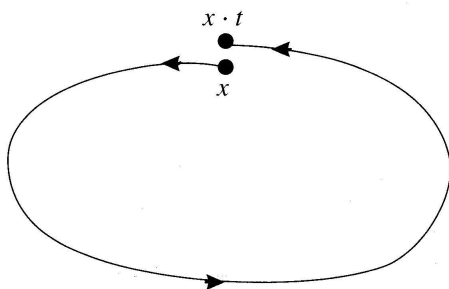
Let us mention the following example illustrating the definition of R_ξ . Consider a smooth flow on a closed manifold M whose chain recurrent set R consists of finitely many rest points and periodic orbits. Given a cohomology class $\xi \in H^1(M; \mathbf{R})$, the chain recurrent set R_ξ is the union of all the rest points and of those periodic orbits whose homology classes $z \in H_1(M; \mathbf{Z})$ satisfy $\langle \xi, z \rangle = 0$.

In general, if the homology class $z \in H_1(M; \mathbf{Z})$ of a periodic orbit satisfies $\langle \xi, z \rangle = 0$ then the orbit belongs to R_ξ . However, it may happen that the points of a periodic orbit belong to R_ξ although $\langle \xi, z \rangle \neq 0$; such an example is described in [6], example after Definition 5.

A different definition of R_ξ which does not use the covering space \tilde{M}_ξ can be found in [6].

To state our main result we also need the following notion.

A (δ, T) -cycle of the flow Φ is defined as a pair (x, t) , where $x \in M$ and $t > T$ such that $d(x, x \cdot t) < \delta$. If δ is small enough then any (δ, T) -cycle determines in a canonical way a unique homology class $z \in H_1(M; \mathbf{Z})$ which



is represented by the flow trajectory from x to $x \cdot t$ followed by a “short” arc connecting $x \cdot t$ with x . See [6].

6. THEOREM

THEOREM 1. *Let V be a smooth vector field on a smooth closed manifold M . Denote by $\Phi: M \times \mathbf{R} \rightarrow M$ the flow generated by V . Let $\xi \in H^1(M; \mathbf{R})$ be a cohomology class such that the restriction $\xi|_{R_\xi}$, viewed as a Čech cohomology class $\xi|_{R_\xi} \in \check{H}^1(R_\xi; \mathbf{R})$, vanishes. Then the following properties of ξ are equivalent:*

(I) *There exists a smooth Lyapunov 1-form for (Φ, R_ξ) in the cohomology class ξ and the subset C_ξ is closed.*

(II) *For any Riemannian metric on M there exist $\delta > 0$ and $T > 1$ such that the homology class $z \in H_1(M; \mathbf{Z})$ associated with an arbitrary (δ, T) -cycle (x, t) of the flow, with $x \in C_\xi$, satisfies $\langle \xi, z \rangle \leq -1$.*

(III) *The subset C_ξ is closed and there exists a constant $\eta > 0$ such that for any Φ -invariant positive Borel measure μ on M the asymptotic cycle $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$ satisfies*

$$(6.1) \quad \langle \xi, \mathcal{A}_\mu \rangle \leq -\eta \cdot \mu(C_\xi).$$

(IV) *The subset C_ξ is closed and for any Φ -invariant positive Borel measure μ on X with $\mu(C_\xi) > 0$, the asymptotic cycle $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$ satisfies*

$$(6.2) \quad \langle \xi, \mathcal{A}_\mu \rangle < 0.$$

The main point of this result is that it gives sufficient homological conditions for the existence of a Lyapunov 1-form in the cohomology class ξ .

Condition (6.1) can be reformulated using the notion of a quasi-regular point. Recall that $x \in X$ is a *quasi-regular* point of the flow $\Phi: X \times \mathbf{R} \rightarrow X$ if for any continuous function $f: X \rightarrow \mathbf{R}$ the limit

$$(6.3) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(x \cdot s) ds$$

exists. It follows from the ergodic theorem that the subset $Q \subset X$ of all quasi-regular points has full measure with respect to any Φ -invariant positive Borel measure on X , see [11], p. 106. From the Riesz representation theorem, see e.g. [15], p. 256, one deduces that for any quasi-regular point $x \in Q$ there exists a unique positive flow-invariant Borel measure μ_x with $\mu_x(X) = 1$ satisfying

$$(6.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(x \cdot s) ds = \int_X f d\mu_x$$

for any continuous function f . We use below the well-known fact that any positive, Φ -invariant Borel measure μ with $\mu(X) = 1$ belongs to the weak* closure of the convex hull of the set of measures μ_x , $x \in Q$, see [11], p. 108.

If the subset $C_\xi \subset X$ is closed, and hence compact, one can apply the above mentioned facts to the restriction of the flow to C_ξ . Let ω be an arbitrary smooth closed 1-form lying in the cohomology class ξ . For any quasi-regular point $x \in C_\xi$ of the flow $\Phi|_{C_\xi}$ one has

$$(6.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_x^{x \cdot t} \omega = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \iota_V(\omega)(x \cdot s) ds = \int_M \iota_V(\omega) d\mu_x = \langle \xi, \mathcal{A}_{\mu_x} \rangle.$$

We therefore conclude that condition (III) is equivalent to:

(III') *The subset C_ξ is closed and there exists a constant $\eta > 0$ such that for any quasi-regular point $x \in C_\xi$,*

$$(6.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_x^{x \cdot t} \omega \leq -\eta,$$

where ω is an arbitrary closed 1-form in the class ξ .

The value of the limit (6.6) is independent of the choice of a closed 1-form ω ; the only requirement is that ω lies in the cohomology class ξ .

In the special case $\xi = 0$ the set C_ξ is empty and $R = R_\xi$. The above statement then reduces to the following well-known theorem of C. Conley – see [1] and [18], Theorem 3.14:

PROPOSITION 2 (C. Conley). *Let V be a smooth vector field on a smooth closed manifold M . Denote by $\Phi: M \times \mathbf{R} \rightarrow M$ the flow generated by V and by R the chain recurrent set of Φ . Then there exists a smooth Lyapunov function $L: M \rightarrow \mathbf{R}$ for (Φ, R) . This means that $V(L) < 0$ on $M - R$ and $dL = 0$ pointwise on R .*

Proposition 2 is used in the proof of Theorem 1. As we could not find a proof of this statement in the literature we present one in the appendix.

Next we state a simple corollary of Theorem 1.

COROLLARY 1. *Let $\Phi: M \times \mathbf{R} \rightarrow M$ be a smooth flow on a closed manifold M . Any de Rham cohomology class $\xi \in H^1(M; \mathbf{R})$ satisfying*

$$\xi|_R = 0 \in \check{H}^1(M; \mathbf{R}),$$

where $R = R(\Phi)$ denotes the chain recurrent set of the flow, contains a Lyapunov 1-form ω for (Φ, R) .

We emphasize that vanishing $\xi|_R = 0$ is supposed to happen in the Čech cohomology. Corollary 1 follows directly from Theorem 1 since under the assumption $\xi|_R = 0$ the set R_ξ coincides with R and so the set C_ξ is empty. Corollary 1 admits also a simple proof independent of Theorem 1 based on Conley's Theorem (Proposition 2 above).

7. EXAMPLES

Here we describe a class of examples of flows for which there exists a cohomology class ξ satisfying all the conditions of Theorem 1.

Let M be a closed smooth manifold with a smooth vector field v . Let $\Psi: M \times \mathbf{R} \rightarrow M$ be the flow of v . Assume that the chain recurrent set $R(\Psi)$ is a union of two disjoint closed sets, $R(\Psi) = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$. With these data we will construct a flow Φ on

$$X = M \times S^1$$

such that $R_\xi(\Phi) = R_1 \times S^0$, $C_\xi = R_2 \times S^1$. Here $\xi \in H^1(X; \mathbf{Z})$ denotes the de Rham cohomology class of the 1-form $-d\theta$ where $\theta \in [0, 2\pi]$ denotes the angle coordinate on the circle S^1 . $S^0 \subset S^1$ is a two-point set.

We will need two vector fields w_1 and w_2 on S^1 , $w_1 = \cos(\theta) \cdot \frac{\partial}{\partial \theta}$ and $w_2 = \frac{\partial}{\partial \theta}$. The field w_1 has two zeros $\{p_1, p_2\} = S^0 \subset S^1$ corresponding to the angles $\theta = \pi/2$ and $\theta = 3\pi/2$.

Let $f_i: M \rightarrow [0, 1]$, where $i = 1, 2$, be two smooth functions having disjoint supports and satisfying $f_1|_{R_1} = 1$, $f_2|_{R_2} = 1$.

Consider the flow $\Phi: X \times \mathbf{R} \rightarrow X$ determined by the vector field

$$V = v + f_1 w_1 + f_2 w_2.$$

Any trajectory of V has the form $(\gamma(t), \theta(t))$, where $\dot{\gamma}(t) = v(\gamma(t))$, i.e. $\gamma(t)$ is a trajectory of v . It follows that the chain recurrent set of V is contained in $R(\Psi) \times S^1$. Over R_1 we have the vertical vector field w_1 along the circle which has two points $S^0 \subset S^1$ as its chain recurrent set. Over R_2 we have the vertical vector field w_2 which has all of S^1 as the chain recurrent set. We see that $R_1 \times S^0 = R_\xi(\Phi)$, $R_2 \times S^1 = C_\xi$. Hence

$$\xi|_{R_\xi} = 0$$

and C_ξ is closed. One easily checks that condition (III) of Theorem 1 (and hence the other conditions as well) is satisfied.

Further examples can be found in section 7 of our paper [6].

8. PROOF OF THEOREM 1

The implication (I) \Rightarrow (II) follows from the proof of Proposition 4 in [6].

(II) \Rightarrow (III). By [6], Theorem 2, the set C_ξ is closed. Now we want to show that the inequality (6.2) is satisfied for any positive Φ -invariant Borel measure μ on X with $\mu(C_\xi) > 0$. Fix a closed 1-form ω in the cohomology class ξ . By Lemma 6 from [6], there exist constants $\alpha > 0$ and $\beta > 0$ such that for any $x \in C_\xi$ and $t > 0$, one has

$$\int_x^{x \cdot t} \omega \leq -\alpha t + \beta.$$

Set $t_0 = 2\beta/\alpha$. Then for any $x \in C_\xi$ and $t \geq t_0$ we have

$$(8.1) \quad \frac{1}{t} \int_x^{x \cdot t} \omega \leq -\frac{\alpha}{2}.$$

With any quasi-regular point $\hat{x} \in C_\xi$ one associates in a canonical way a positive Φ -invariant Borel measure $\mu_{\hat{x}}$ on C_ξ , see above. It has the property that

$$(8.2) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_x^{x \cdot t} \omega = \int_M \iota_V(\omega) d\mu_x.$$

From (8.1) and (8.2) one obtains

$$(8.3) \quad \langle \xi, \mathcal{A}_{\mu_x} \rangle \leq -\frac{\alpha}{2} < 0$$

for any quasi-regular point $x \in C_\xi$. According to [11], p.108, any positive Φ -invariant Borel measure μ with $\mu(M) = \mu(C_\xi) = 1$ belongs to the weak* closure of the convex hull of the set of measures $\{\mu_x; x \in C_\xi \text{ is quasi-regular}\}$; hence

$$(8.4) \quad \langle \xi, \mathcal{A}_\mu \rangle \leq -\frac{\alpha}{2} < 0.$$

It is well known that every (finite) positive Φ -invariant Borel measure is supported on $R = R_\xi \cup C_\xi$, see e.g. [13], Proposition 4.1.18, p.141. As R_ξ and C_ξ are closed and flow-invariant we may write $\mu = \mu_1 + \mu_2$ where μ_1, μ_2 are Φ -invariant and μ_1 is supported on R_ξ , while μ_2 is supported on C_ξ . It follows from (8.4) that $\langle \xi, \mathcal{A}_{\mu_2} \rangle \leq -\frac{\alpha}{2} \cdot \mu_2(C_\xi)$. Further, we claim that $\langle \xi, \mathcal{A}_{\mu_1} \rangle = 0$ for the following reason. Since $\xi|_{R_\xi} = 0$ (as a Čech cohomology class), for any smooth closed 1-form ω on M representing ξ there exists a smooth function f defined on an open neighborhood of R_ξ such that $\omega = df$ near R_ξ . Then we obtain

$$(8.5) \quad \langle \xi, \mathcal{A}_{\mu_1} \rangle = \int_M \iota_V(\omega) d\mu_1 = \int_{R_\xi} \iota_V(\omega) d\mu_1 = \int_{R_\xi} V(f) d\mu_1 = 0.$$

The last equality holds since the measure μ_1 is Φ -invariant (see e.g. [16], Theorem on page 277). Finally, as $\mathcal{A}_\mu = \mathcal{A}_{\mu_1} + \mathcal{A}_{\mu_2}$ we see that $\langle \xi, \mathcal{A}_\mu \rangle \leq -\eta \cdot \mu(C_\xi)$ with $\eta = \alpha/2$, which completes the proof of (II) \Rightarrow (III).

The implication (III) \Rightarrow (IV) is obvious.

We are left to show the implication (IV) \Rightarrow (I). Our argument uses the technique of Schwartzman [16]. It is to show that under the conditions (IV) there exists a smooth Lyapunov 1-form for (Φ, R_ξ) in the class ξ . In a first step we prove that there exists a smooth, closed 1-form ω_1 in the class ξ so that $\iota_V(\omega_1) < 0$ on C_ξ . To this end, denote by $\mathcal{D} \subset C^0(M)$ the space of functions

$$\mathcal{D} = \{V(f); f: M \rightarrow \mathbf{R} \text{ is smooth}\}$$

and by \mathcal{C}^- the convex cone in $C^0(M)$ consisting of all functions $f \in C^0(M)$ with

$$f(x) < 0 \quad \text{for all } x \in C_\xi.$$

As C_ξ is compact, the cone \mathcal{C}^- is open in the Banach space $C^0(M)$ of continuous functions on M , endowed with the usual supremum norm. Choose an arbitrary smooth, closed 1-form ω in the class ξ . Assume that $\mathcal{C}^- \cap (\iota_V(\omega) + \mathcal{D}) = \emptyset$. It then follows from the Hahn–Banach Theorem (cf. [15], p.58) that there exists a continuous linear functional $\Lambda: C^0(M) \rightarrow \mathbf{R}$ such that

$$\Lambda|_{\iota_V(\omega) + \mathcal{D}} \geq 0 \quad \text{and} \quad \Lambda|_{\mathcal{C}^-} < 0.$$

Since $\iota_V(\omega) + \mathcal{D}$ is an affine subspace and Λ is bounded on it from below, we obtain that Λ restricted to \mathcal{D} vanishes. According to the Riesz representation theorem (cf. [12]), there exists a Borel measure μ on M so that

$$\Lambda(f) = \int_M f d\mu$$

for any $f \in C^0(M)$. By Theorem [16], p.277, the condition $\Lambda|_{\mathcal{D}} = 0$ implies that μ is Φ -invariant. On the other hand, $\Lambda|_{\mathcal{C}^-} < 0$ implies that $\mu|_{C_\xi} > 0$.

Denote by $\chi: M \rightarrow \mathbf{R}$ the characteristic function of C_ξ and let $\nu = \chi \cdot \mu$. As C_ξ is Φ -invariant ν is a Φ -invariant Borel measure and (unlike, possibly, μ) is positive. Note that $\mu - \nu$ is a Φ -invariant Borel measure supported on R_ξ (again using that any Φ -invariant measure is supported on $R = R_\xi \cup C_\xi$). Thus, it follows from our assumption $\xi|_{R_\xi} = 0$, by the same argument which led to (8.5), that

$$\langle \xi, \mathcal{A}_{\mu - \nu} \rangle = 0.$$

Since $\mathcal{A}_{\mu - \nu} = \mathcal{A}_\mu - \mathcal{A}_\nu$ we find

$$\langle \xi, \mathcal{A}_\nu \rangle = \langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega) d\mu = \Lambda(f) \geq 0$$

where $f = \iota_V(\omega)$, contradicting condition (6.2). This means that the intersection $\mathcal{C}^- \cap (\iota_V(\omega) + \mathcal{D})$ cannot be empty, i.e. there exists a smooth function $g: M \rightarrow \mathbf{R}$ so that the smooth closed 1-form $\omega_1 = \omega + dg$ is in the class ξ and satisfies

$$\iota_V(\omega_1) < 0 \quad \text{on} \quad C_\xi.$$

This completes the first step of the proof.

To finish the argument, we now adjust ω_1 on the complement of C_ξ so that the resulting form is a Lyapunov 1-form for (Φ, R_ξ) . As $\iota_V(\omega_1) < 0$ on C_ξ and C_ξ is compact, there is some open neighborhood W_1 of C_ξ such that $W_1 \cap R_\xi = \emptyset$ and $\iota_V(\omega_1) < 0$ on W_1 . Since $\xi|_{R_\xi} = 0$, there exists an open neighborhood W_2 of R_ξ such that $W_1 \cap W_2 = \emptyset$ and a smooth function $g: M \rightarrow \mathbf{R}$ such that $\omega_1|_{W_2} = dg$ and $dg|_{W_1} = 0$. By Proposition 2 there exists a smooth Lyapunov function $L: M \rightarrow \mathbf{R}$ for (Φ, R) . Now consider

$$(8.6) \quad \omega_2 = \omega_1 - dg + \lambda dL,$$

where $\lambda > 0$ remains to be chosen. Clearly, the form ω_2 is smooth and closed and represents the class ξ . For any $\lambda > 0$ it satisfies $\omega_2|_{W_2} = d(\lambda L)$, because $\omega_1 - dg$ vanishes on this set. In particular, ω_2 has property $(\Lambda 2)$ of a Lyapunov 1-form for the pair (Φ, R_ξ) . Note also that for all positive λ we have $\iota_V(\omega_2) < 0$ on W_1 (by the construction of W_1) and on $W_2 - R_\xi$ because $\omega_1 - dg$ vanishes there, whereas $V(L) < 0$. As the complement of $W_1 \cup W_2$ is compact and disjoint from R ,

$$1 < \lambda_0 := 1 + \sup_{x \notin W_1 \cup W_2} \frac{|\iota_V(\omega_1 - dg)|}{|V(L)|} < \infty,$$

and $\iota_V(\omega_2) < 0$ on $M - R_\xi$ for all $\lambda \geq \lambda_0$, showing that for such choices of λ the form ω_2 also has property $(\Lambda 1)$ of a Lyapunov 1-form for (Φ, R_ξ) . This completes the proof of the implication $(IV) \Rightarrow (I)$ and hence the proof of Theorem 1. \square

APPENDIX: PROOF OF PROPOSITION 2

Recall from [1, II 6.2.A] the alternative characterization of the chain recurrent set R as

$$R = \bigcap \{A \cup A^* \mid (A, A^*) \text{ is an attractor-repeller pair}\}.$$

Here a closed, flow-invariant subset $A \subset M$ is called an attractor if it admits a neighborhood U such that A is the maximal flow-invariant subset in the closure of $U \cdot [0, \infty)$. The dual repeller A^* is the set of all points $x \in M$ whose forward limit set is disjoint from A (cf. [1, II 5.1]). Equivalently, (A, A^*) is an attractor-repeller pair if and only if both A and A^* are closed flow invariant subsets of M and the forward (resp. backward) limit set of every point $x \notin A \cup A^*$ is contained in A (resp. A^*) – see [14, Prop. 1.4].

As M is a closed manifold and hence separable, the number of distinct attractor-repeller pairs is at most countable (cf. [1, II 6.4.A]). Let $\{(A_n, A_n^*)\}_{n \geq 1}$ be some enumeration. For each $n \geq 1$, the construction of Robbin and Salamon (Prop. 1.4. of [14] and the remark following it) yields a smooth function $f_n: M \rightarrow [0, 1]$ with $f_n^{-1}(0) = A_n$, $f_n^{-1}(1) = A_n^*$ and $df_n(V) < 0$ on the complement of $A_n \cup A_n^*$. Let c_n be positive constants such that in a fixed finite atlas of charts all partial derivatives of f_n of order $\leq n$ are bounded pointwise in absolute value by c_n . Then

$$L(x) := \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n c_n}$$

is a smooth function having the required properties. In particular, as for any $n \geq 1$ the differential df_n vanishes on $A_n \cup A_n^*$, the differential of L vanishes on R . \square

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