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## SMOOTH LYAPUNOV 1-FORMS

by M. FARBER \*), T. KAPPELER, J. LATSCHEV †) and E. ZEHNDER

**ABSTRACT.** We find conditions which guarantee that a given flow  $\Phi$  on a closed smooth manifold  $M$  admits a smooth Lyapunov 1-form  $\omega$  lying in a prescribed de Rham cohomology class  $\xi \in H^1(M; \mathbf{R})$ . These conditions are formulated in terms of Schwartzman's asymptotic cycles  $\mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$  of the flow.

### 1. INTRODUCTION

C. Conley [1, 2] showed that any continuous flow  $\Phi: X \times \mathbf{R} \rightarrow X$  on a compact metric space  $X$  “decomposes” into a chain recurrent flow and a gradient-like flow. More precisely, he proved the existence of a continuous function  $L: X \rightarrow \mathbf{R}$  which (i) decreases along any orbit of the flow in the complement  $X - R$  of the chain recurrent set  $R \subset X$  of  $\Phi$  and (ii) is constant on the connected components of  $R$ . Such a function  $L$  is called a *Lyapunov function* for  $\Phi$ . This existence result plays a fundamental role in Conley's program of understanding general flows as collections of isolated invariant sets linked by heteroclinic orbits.

A more general notion of a *Lyapunov 1-form* was introduced in paper [5]. Lyapunov 1-forms, as compared to Lyapunov functions, allow one to go one step further and to analyze the flow within the chain-recurrent set  $R$  as well. Lyapunov 1-forms provide an important tool in applying methods of homotopy theory to dynamical systems. In the recent papers [4], [5] a generalization of the Lusternik–Schnirelman theory was constructed which applies to flows admitting Lyapunov 1-forms.

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The problem of existence of Lyapunov 1-forms was addressed in our recent preprint [6], where we worked in the category of compact metric spaces, continuous flows and continuous closed 1-forms. In the present paper we study the smooth version of the problem: we construct smooth Lyapunov 1-forms for smooth flows on smooth manifolds. We use Schwartzman's asymptotic cycles to formulate a necessary condition for the existence of Lyapunov 1-forms in a given cohomology class. We also show that under an additional assumption this condition is equivalent to the homological condition introduced in our previous paper [6].

## 2. DEFINITION

Let  $V$  be a smooth vector field on a smooth manifold  $M$ . Assume that  $V$  generates a continuous flow  $\Phi: M \times \mathbf{R} \rightarrow M$  and  $Y \subset M$  is a closed, flow-invariant subset.

**DEFINITION 1.** A smooth closed 1-form  $\omega$  on  $M$  is called a *Lyapunov 1-form* for the pair  $(\Phi, Y)$  if it has the following properties:

- ( $\Lambda$ 1) The function  $\iota_V(\omega) = \omega(V)$  is negative on  $M - Y$ ;
- ( $\Lambda$ 2) There exists a smooth function  $f: U \rightarrow \mathbf{R}$  defined on an open neighborhood  $U$  of  $Y$  such that

$$\omega|_U = df \quad \text{and} \quad df|_Y = 0.$$

The above definition is a modification of the notion of a Lyapunov 1-form introduced in section 6 of [5]. The definition of [5] requires that  $Y$  consists of finitely many points and the vector field  $V$  is locally a gradient of  $\omega$  with respect to a Riemannian metric.

Definition 1 can also be compared with the definition of a Lyapunov 1-form in the continuous setting which was introduced in [6]. Condition ( $\Lambda$ 1) above is slightly stronger than condition (L1) of Definition 1 in [6]. Condition ( $\Lambda$ 2) is similar to condition (L2) of Definition 1 from [6] although they are not equivalent.

There are several natural alternatives for condition ( $\Lambda$ 2). One of them is:

- ( $\Lambda$ 2') The 1-form  $\omega$ , viewed as a map  $\omega: M \rightarrow T^*(M)$ , vanishes on  $Y$ .

It is clear that ( $\Lambda$ 2) implies ( $\Lambda$ 2'). We can show that the converse is true under some additional assumptions:

LEMMA 1. *If the de Rham cohomology class  $\xi$  of  $\omega$  is integral,  $\xi = [\omega] \in H^1(M; \mathbf{Z})$ , then the conditions  $(\Lambda 2')$  and  $(\Lambda 2)$  are equivalent.*

*Proof.* Clearly we only need to show that  $(\Lambda 2')$  implies  $(\Lambda 2)$ . Since  $\xi$  is integral there exists a smooth map  $\phi: M \rightarrow S^1$  such that  $\omega = \phi^*(d\theta)$ , where  $d\theta$  is the standard angular 1-form on the circle  $S^1$ . Let  $\alpha \in S^1$  be a regular value of  $\phi$ . Assuming that  $(\Lambda 2')$  holds it then follows that  $U = M - \phi^{-1}(\alpha)$  is an open neighborhood of  $Y$ . Clearly  $\omega|_U = df$  where  $f: U \rightarrow \mathbf{R}$  is a smooth function which is related to  $\phi$  by  $\phi(x) = \exp(if(x))$  for any  $x \in U$ . Hence  $(\Lambda 2)$  holds.

LEMMA 2. *The conditions  $(\Lambda 2')$  and  $(\Lambda 2)$  are equivalent if  $Y$  is an Euclidean Neighborhood Retract (ENR).*

*Proof.* Again, we only have to establish  $(\Lambda 2') \Rightarrow (\Lambda 2)$ . Since  $Y$  is an ENR it admits an open neighbourhood  $U \subset M$  such that the inclusion  $i_U: U \rightarrow M$  is homotopic to  $i_Y \circ r$ , where  $i_Y: Y \rightarrow M$  is the inclusion and  $r: U \rightarrow Y$  is a retraction (see [3], chapter 4, §8, Corollary 8.7). Pick a base point  $x_j$  in every path-connected component  $U_j$  of  $U$  and define a smooth function  $f_j: U_j \rightarrow \mathbf{R}$  by

$$f_j(x) = \int_{x_j}^x \omega, \quad x \in U_j.$$

The latter integral is independent of the choice of the integration path in  $U_j$  connecting  $x_j$  with  $x$ . This claim is equivalent to the vanishing of the integral  $\int_\gamma \omega$  for any closed loop  $\gamma$  lying in  $U$ . To show this we apply the retraction to see that  $\gamma$  is homotopic in  $M$  to the loop  $\gamma_1 = r \circ \gamma$ , which lies in  $Y$ ; thus we obtain  $\int_\gamma \omega = \int_{\gamma_1} \omega = 0$  because of  $(\Lambda 2')$ . It is clear that the functions  $f_j$  together determine a smooth function  $f: U \rightarrow \mathbf{R}$  with  $df = \omega|_U$ .

A class of interesting examples can be obtained as follows. Let  $\omega$  be a smooth closed 1-form on a closed Riemannian manifold  $M$ . Consider the negative gradient vector field  $V$  of  $\omega$ , i.e.  $\langle V, X \rangle = -\omega(X)$  for any vector field  $X$  on  $M$  where  $\langle \cdot, \cdot \rangle$  denotes the Riemannian metric. Denote by  $\Phi$  the flow induced by the vector field  $V$  and by  $Y$  the set of zeros of  $\omega$ . Then clearly conditions  $(\Lambda 1)$  and  $(\Lambda 2')$  are satisfied. If either the cohomology class of  $\omega$  is integral or  $Y$  is an ENR then (by the two Lemmas above)  $\omega$  is a Lyapunov 1-form for the pair  $(\Phi, Y)$ .

Our main goal in this paper is to find topological conditions which guarantee that for a given vector field  $V$  on  $M$  there exists a Lyapunov 1-form  $\omega$  lying in a prescribed cohomology class  $\xi \in H^1(M; \mathbf{R})$ .

### 3. ASYMPTOTIC CYCLES OF SCHWARTZMAN

Let  $M$  be a closed smooth manifold and let  $V$  be a smooth vector field. Let  $\Phi: M \times \mathbf{R} \rightarrow M$  be the flow generated by  $V$ .

Consider a Borel measure  $\mu$  on  $M$  which is invariant under  $\Phi$ . According to S. Schwartzman [16], these data determine a real homology class

$$\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$$

called *the asymptotic cycle of the flow  $\Phi$  corresponding to the measure  $\mu$* . The class  $\mathcal{A}_\mu$  is defined as follows. For a de Rham cohomology class  $\xi \in H^1(M; \mathbf{R})$  the evaluation  $\langle \xi, \mathcal{A}_\mu \rangle \in \mathbf{R}$  is given by the integral

$$(3.1) \quad \langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega) d\mu,$$

where  $\omega$  is a closed 1-form in the class  $\xi$ . Note that  $\langle \xi, \mathcal{A}_\mu \rangle$  is well-defined, i.e. it depends only on the cohomology class  $\xi$  of  $\omega$ , see [16], p. 277. Indeed, replacing  $\omega$  by  $\omega' = \omega + df$ , where  $f: M \rightarrow \mathbf{R}$  is a smooth function, the integral in (3.1) gets changed by the quantity

$$(3.2) \quad \int_M V(f) d\mu = \lim_{s \rightarrow 0} \frac{1}{s} \int_M \{f(x \cdot s) - f(x)\} d\mu(x).$$

Here  $V(f)$  denotes the derivative of  $f$  in the direction of the vector field  $V$  and  $x \cdot s$  stands for the flow  $\Phi(x, s)$  of the vector field  $V$ . Since the measure  $\mu$  is flow invariant, the integral on the RHS of (3.2) vanishes for any  $f$ . It is clear that the RHS of (3.1) is a linear function of  $\xi \in H^1(M; \mathbf{R})$ . Hence there exists a unique real homology class  $\mathcal{A}_\mu \in H_1(M; \mathbf{R})$  which satisfies (3.1) for all  $\xi \in H^1(M; \mathbf{R})$ .

### 4. NECESSARY CONDITIONS

We consider the flow  $\Phi$  as being fixed and we vary the invariant measure  $\mu$ . As the class  $\mathcal{A}_\mu \in H_1(M; \mathbf{R})$  depends linearly on  $\mu$ , the set of asymptotic cycles  $\mathcal{A}_\mu$  corresponding to all  $\Phi$ -invariant positive measures  $\mu$  forms a convex cone in the vector space  $H_1(M; \mathbf{R})$ .

PROPOSITION 1. *Assume that there exists a Lyapunov 1-form for  $(\Phi, Y)$  lying in a cohomology class  $\xi \in H^1(M; \mathbf{R})$ . Then*

$$(4.1) \quad \langle \xi, \mathcal{A}_\mu \rangle \leq 0$$

*for any  $\Phi$ -invariant positive Borel measure  $\mu$  on  $M$ ; equality in (4.1) takes place if and only if the complement of  $Y$  has measure zero. Further, the restriction of  $\xi$  to  $Y$ , viewed as a Čech cohomology class*

$$\xi|_Y \in \check{H}^1(Y; \mathbf{R})$$

*vanishes,  $\xi|_Y = 0$ .*

*Proof.* Let  $\omega$  be a Lyapunov 1-form for  $(\Phi, Y)$  lying in the class  $\xi$ . According to Definition 1, the function  $\iota_V(\omega)$  is negative on  $M - Y$  and vanishes on  $Y$ . We obtain that the integral

$$\int_M \iota_V(\omega) d\mu = \langle \xi, \mathcal{A}_\mu \rangle$$

is nonpositive.

Assuming  $\mu(M - Y) > 0$ , we find a compact  $K \subset M - Y$  with  $\mu(K) > 0$ ; this follows from the Theorem of Riesz – see e.g. [12], Theorem 2.3(iv), p. 256. There is a constant  $\epsilon > 0$  such that  $\iota_V(\omega)|_K \leq -\epsilon$ . Therefore, one has

$$\int_M \iota_V(\omega) d\mu \leq -\epsilon \mu(K) < 0.$$

Hence, the value  $\langle \xi, \mathcal{A}_\mu \rangle$  is strictly negative if the measure  $\mu$  is not supported in  $Y$ .

To prove the second statement we observe (see [19]) that the Čech cohomology  $\check{H}^1(Y; \mathbf{R})$  equals the direct limit of the singular cohomology

$$\check{H}^1(Y; \mathbf{R}) = \lim_{W \supset Y} H^1(W; \mathbf{R}),$$

where  $W$  runs over open neighborhoods of  $Y$ . It is clear in view of condition (A2) that  $\xi|_U = 0 \in H^1(U; \mathbf{R})$  (by the de Rham theorem). Hence the result follows.

## 5. CHAIN-RECURRENT SET $R_\xi$

Given a flow  $\Phi$ , our aim is to construct a Lyapunov 1-form  $\omega$  for a pair  $(\Phi, Y)$  lying in a given cohomology class  $\xi \in H^1(M; \mathbf{R})$ . A natural candidate

for  $Y$  is the subset  $R_\xi = R_\xi(\Phi)$  of the chain-recurrent set  $R = R(\Phi)$  which was defined in [6]. For convenience of the reader we briefly recall the definition.

Fix a Riemannian metric on  $M$  and denote by  $d$  the corresponding distance function. Given any  $\delta > 0$ ,  $T > 1$ , a  $(\delta, T)$ -chain from  $x \in M$  to  $y \in M$  is a finite sequence  $x_0 = x, x_1, \dots, x_N = y$  of points in  $M$  and numbers  $t_1, \dots, t_N \in \mathbf{R}$  such that  $t_i \geq T$  and  $d(x_{i-1} \cdot t_i, x_i) < \delta$  for all  $1 \leq i \leq N$ . Here we use the notation  $\Phi(x, t) = x \cdot t$ . The chain recurrent set  $R = R(\Phi)$  of the flow  $\Phi$  is defined as the set of all points  $x \in M$  such that for any  $\delta > 0$  and  $T > 1$  there exists a  $(\delta, T)$ -chain starting and ending at  $x$ . The chain recurrent set is closed and invariant under the flow.

Given a cohomology class  $\xi \in H^1(M; \mathbf{R})$  there is a natural covering space  $p_\xi: \tilde{M}_\xi \rightarrow M$  associated with  $\xi$ . A closed loop  $\gamma: [0, 1] \rightarrow M$  lifts to a closed loop in  $\tilde{M}_\xi$  if and only if the value of the cohomology class  $\xi$  on the homology class  $[\gamma] \in H_1(M; \mathbf{Z})$  vanishes,  $\langle \xi, [\gamma] \rangle = 0$ . See [19].

The flow  $\Phi$  lifts uniquely to a flow  $\tilde{\Phi}$  on the covering  $\tilde{M}_\xi$ . Consider the chain recurrent set  $R(\tilde{\Phi}) \subset \tilde{M}_\xi$  of the lifted flow and denote by  $R_\xi = p_\xi(R(\tilde{\Phi})) \subset M$  its projection onto  $M$ . The set  $R_\xi$  is referred to as the chain recurrent set associated to the cohomology class  $\xi$ . It is clear that  $R_\xi$  is a closed and  $\Phi$ -invariant subset of  $R$ . We denote by  $C_\xi$  the complement of  $R_\xi$  in  $R$ ,

$$C_\xi = R - R_\xi.$$

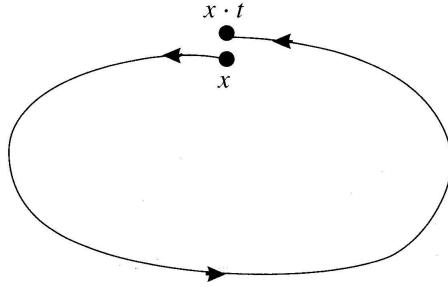
Let us mention the following example illustrating the definition of  $R_\xi$ . Consider a smooth flow on a closed manifold  $M$  whose chain recurrent set  $R$  consists of finitely many rest points and periodic orbits. Given a cohomology class  $\xi \in H^1(M; \mathbf{R})$ , the chain recurrent set  $R_\xi$  is the union of all the rest points and of those periodic orbits whose homology classes  $z \in H_1(M; \mathbf{Z})$  satisfy  $\langle \xi, z \rangle = 0$ .

In general, if the homology class  $z \in H_1(M; \mathbf{Z})$  of a periodic orbit satisfies  $\langle \xi, z \rangle = 0$  then the orbit belongs to  $R_\xi$ . However, it may happen that the points of a periodic orbit belong to  $R_\xi$  although  $\langle \xi, z \rangle \neq 0$ ; such an example is described in [6], example after Definition 5.

A different definition of  $R_\xi$  which does not use the covering space  $\tilde{M}_\xi$  can be found in [6].

To state our main result we also need the following notion.

A  $(\delta, T)$ -cycle of the flow  $\Phi$  is defined as a pair  $(x, t)$ , where  $x \in M$  and  $t > T$  such that  $d(x, x \cdot t) < \delta$ . If  $\delta$  is small enough then any  $(\delta, T)$ -cycle determines in a canonical way a unique homology class  $z \in H_1(M; \mathbf{Z})$  which



is represented by the flow trajectory from  $x$  to  $x \cdot t$  followed by a “short” arc connecting  $x \cdot t$  with  $x$ . See [6].

## 6. THEOREM

**THEOREM 1.** *Let  $V$  be a smooth vector field on a smooth closed manifold  $M$ . Denote by  $\Phi: M \times \mathbf{R} \rightarrow M$  the flow generated by  $V$ . Let  $\xi \in H^1(M; \mathbf{R})$  be a cohomology class such that the restriction  $\xi|_{R_\xi}$ , viewed as a Čech cohomology class  $\xi|_{R_\xi} \in \check{H}^1(R_\xi; \mathbf{R})$ , vanishes. Then the following properties of  $\xi$  are equivalent:*

(I) *There exists a smooth Lyapunov 1-form for  $(\Phi, R_\xi)$  in the cohomology class  $\xi$  and the subset  $C_\xi$  is closed.*

(II) *For any Riemannian metric on  $M$  there exist  $\delta > 0$  and  $T > 1$  such that the homology class  $z \in H_1(M; \mathbf{Z})$  associated with an arbitrary  $(\delta, T)$ -cycle  $(x, t)$  of the flow, with  $x \in C_\xi$ , satisfies  $\langle \xi, z \rangle \leq -1$ .*

(III) *The subset  $C_\xi$  is closed and there exists a constant  $\eta > 0$  such that for any  $\Phi$ -invariant positive Borel measure  $\mu$  on  $M$  the asymptotic cycle  $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$  satisfies*

$$(6.1) \quad \langle \xi, \mathcal{A}_\mu \rangle \leq -\eta \cdot \mu(C_\xi).$$

(IV) *The subset  $C_\xi$  is closed and for any  $\Phi$ -invariant positive Borel measure  $\mu$  on  $M$  with  $\mu(C_\xi) > 0$ , the asymptotic cycle  $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$  satisfies*

$$(6.2) \quad \langle \xi, \mathcal{A}_\mu \rangle < 0.$$

The main point of this result is that it gives sufficient homological conditions for the existence of a Lyapunov 1-form in the cohomology class  $\xi$ .

Condition (6.1) can be reformulated using the notion of a quasi-regular point. Recall that  $x \in X$  is a *quasi-regular* point of the flow  $\Phi: X \times \mathbf{R} \rightarrow X$  if for any continuous function  $f: X \rightarrow \mathbf{R}$  the limit

$$(6.3) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(x \cdot s) ds$$

exists. It follows from the ergodic theorem that the subset  $Q \subset X$  of all quasi-regular points has full measure with respect to any  $\Phi$ -invariant positive Borel measure on  $X$ , see [11], p. 106. From the Riesz representation theorem, see e.g. [15], p. 256, one deduces that for any quasi-regular point  $x \in Q$  there exists a unique positive flow-invariant Borel measure  $\mu_x$  with  $\mu_x(X) = 1$  satisfying

$$(6.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(x \cdot s) ds = \int_X f d\mu_x$$

for any continuous function  $f$ . We use below the well-known fact that any positive,  $\Phi$ -invariant Borel measure  $\mu$  with  $\mu(X) = 1$  belongs to the weak\* closure of the convex hull of the set of measures  $\mu_x$ ,  $x \in Q$ , see [11], p. 108.

If the subset  $C_\xi \subset X$  is closed, and hence compact, one can apply the above mentioned facts to the restriction of the flow to  $C_\xi$ . Let  $\omega$  be an arbitrary smooth closed 1-form lying in the cohomology class  $\xi$ . For any quasi-regular point  $x \in C_\xi$  of the flow  $\Phi|_{C_\xi}$  one has

$$(6.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_x^{x \cdot t} \omega = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \iota_V(\omega)(x \cdot s) ds = \int_M \iota_V(\omega) d\mu_x = \langle \xi, \mathcal{A}_{\mu_x} \rangle.$$

We therefore conclude that condition (III) is equivalent to:

(III') *The subset  $C_\xi$  is closed and there exists a constant  $\eta > 0$  such that for any quasi-regular point  $x \in C_\xi$ ,*

$$(6.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_x^{x \cdot t} \omega \leq -\eta,$$

where  $\omega$  is an arbitrary closed 1-form in the class  $\xi$ .

The value of the limit (6.6) is independent of the choice of a closed 1-form  $\omega$ ; the only requirement is that  $\omega$  lies in the cohomology class  $\xi$ .

In the special case  $\xi = 0$  the set  $C_\xi$  is empty and  $R = R_\xi$ . The above statement then reduces to the following well-known theorem of C. Conley – see [1] and [18], Theorem 3.14:

PROPOSITION 2 (C. Conley). *Let  $V$  be a smooth vector field on a smooth closed manifold  $M$ . Denote by  $\Phi: M \times \mathbf{R} \rightarrow M$  the flow generated by  $V$  and by  $R$  the chain recurrent set of  $\Phi$ . Then there exists a smooth Lyapunov function  $L: M \rightarrow \mathbf{R}$  for  $(\Phi, R)$ . This means that  $V(L) < 0$  on  $M - R$  and  $dL = 0$  pointwise on  $R$ .*

Proposition 2 is used in the proof of Theorem 1. As we could not find a proof of this statement in the literature we present one in the appendix.

Next we state a simple corollary of Theorem 1.

COROLLARY 1. *Let  $\Phi: M \times \mathbf{R} \rightarrow M$  be a smooth flow on a closed manifold  $M$ . Any de Rham cohomology class  $\xi \in H^1(M; \mathbf{R})$  satisfying*

$$\xi|_R = 0 \in \check{H}^1(M; \mathbf{R}),$$

where  $R = R(\Phi)$  denotes the chain recurrent set of the flow, contains a Lyapunov 1-form  $\omega$  for  $(\Phi, R)$ .

We emphasize that vanishing  $\xi|_R = 0$  is supposed to happen in the Čech cohomology. Corollary 1 follows directly from Theorem 1 since under the assumption  $\xi|_R = 0$  the set  $R_\xi$  coincides with  $R$  and so the set  $C_\xi$  is empty. Corollary 1 admits also a simple proof independent of Theorem 1 based on Conley's Theorem (Proposition 2 above).

## 7. EXAMPLES

Here we describe a class of examples of flows for which there exists a cohomology class  $\xi$  satisfying all the conditions of Theorem 1.

Let  $M$  be a closed smooth manifold with a smooth vector field  $v$ . Let  $\Psi: M \times \mathbf{R} \rightarrow M$  be the flow of  $v$ . Assume that the chain recurrent set  $R(\Psi)$  is a union of two disjoint closed sets,  $R(\Psi) = R_1 \cup R_2$  and  $R_1 \cap R_2 = \emptyset$ . With these data we will construct a flow  $\Phi$  on

$$X = M \times S^1$$

such that  $R_\xi(\Phi) = R_1 \times S^0$ ,  $C_\xi = R_2 \times S^1$ . Here  $\xi \in H^1(X; \mathbf{Z})$  denotes the de Rham cohomology class of the 1-form  $-d\theta$  where  $\theta \in [0, 2\pi]$  denotes the angle coordinate on the circle  $S^1$ .  $S^0 \subset S^1$  is a two-point set.

We will need two vector fields  $w_1$  and  $w_2$  on  $S^1$ ,  $w_1 = \cos(\theta) \cdot \frac{\partial}{\partial \theta}$  and  $w_2 = \frac{\partial}{\partial \theta}$ . The field  $w_1$  has two zeros  $\{p_1, p_2\} = S^0 \subset S^1$  corresponding to the angles  $\theta = \pi/2$  and  $\theta = 3\pi/2$ .

Let  $f_i: M \rightarrow [0, 1]$ , where  $i = 1, 2$ , be two smooth functions having disjoint supports and satisfying  $f_1|_{R_1} = 1$ ,  $f_2|_{R_2} = 1$ .

Consider the flow  $\Phi: X \times \mathbf{R} \rightarrow X$  determined by the vector field

$$V = v + f_1 w_1 + f_2 w_2.$$

Any trajectory of  $V$  has the form  $(\gamma(t), \theta(t))$ , where  $\dot{\gamma}(t) = v(\gamma(t))$ , i.e.  $\gamma(t)$  is a trajectory of  $v$ . It follows that the chain recurrent set of  $V$  is contained in  $R(\Psi) \times S^1$ . Over  $R_1$  we have the vertical vector field  $w_1$  along the circle which has two points  $S^0 \subset S^1$  as its chain recurrent set. Over  $R_2$  we have the vertical vector field  $w_2$  which has all of  $S^1$  as the chain recurrent set. We see that  $R_1 \times S^0 = R_\xi(\Phi)$ ,  $R_2 \times S^1 = C_\xi$ . Hence

$$\xi|_{R_\xi} = 0$$

and  $C_\xi$  is closed. One easily checks that condition (III) of Theorem 1 (and hence the other conditions as well) is satisfied.

Further examples can be found in section 7 of our paper [6].

## 8. PROOF OF THEOREM 1

The implication (I)  $\Rightarrow$  (II) follows from the proof of Proposition 4 in [6].

(II)  $\Rightarrow$  (III). By [6], Theorem 2, the set  $C_\xi$  is closed. Now we want to show that the inequality (6.2) is satisfied for any positive  $\Phi$ -invariant Borel measure  $\mu$  on  $X$  with  $\mu(C_\xi) > 0$ . Fix a closed 1-form  $\omega$  in the cohomology class  $\xi$ . By Lemma 6 from [6], there exist constants  $\alpha > 0$  and  $\beta > 0$  such that for any  $x \in C_\xi$  and  $t > 0$ , one has

$$\int_x^{x+t} \omega \leq -\alpha t + \beta.$$

Set  $t_0 = 2\beta/\alpha$ . Then for any  $x \in C_\xi$  and  $t \geq t_0$  we have

$$(8.1) \quad \frac{1}{t} \int_x^{x+t} \omega \leq -\frac{\alpha}{2}.$$

With any quasi-regular point  $x \in C_\xi$  one associates in a canonical way a positive  $\Phi$ -invariant Borel measure  $\mu_x$  on  $C_\xi$ , see above. It has the property that

$$(8.2) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_x^{x+t} \omega = \int_M \iota_V(\omega) d\mu_x.$$

From (8.1) and (8.2) one obtains

$$(8.3) \quad \langle \xi, \mathcal{A}_{\mu_x} \rangle \leq -\frac{\alpha}{2} < 0$$

for any quasi-regular point  $x \in C_\xi$ . According to [11], p. 108, any positive  $\Phi$ -invariant Borel measure  $\mu$  with  $\mu(M) = \mu(C_\xi) = 1$  belongs to the weak\* closure of the convex hull of the set of measures  $\{\mu_x; x \in C_\xi \text{ is quasi-regular}\}$ ; hence

$$(8.4) \quad \langle \xi, \mathcal{A}_\mu \rangle \leq -\frac{\alpha}{2} < 0.$$

It is well known that every (finite) positive  $\Phi$ -invariant Borel measure is supported on  $R = R_\xi \cup C_\xi$ , see e.g. [13], Proposition 4.1.18, p. 141. As  $R_\xi$  and  $C_\xi$  are closed and flow-invariant we may write  $\mu = \mu_1 + \mu_2$  where  $\mu_1, \mu_2$  are  $\Phi$ -invariant and  $\mu_1$  is supported on  $R_\xi$ , while  $\mu_2$  is supported on  $C_\xi$ . It follows from (8.4) that  $\langle \xi, \mathcal{A}_{\mu_2} \rangle \leq -\frac{\alpha}{2} \cdot \mu_2(C_\xi)$ . Further, we claim that  $\langle \xi, \mathcal{A}_{\mu_1} \rangle = 0$  for the following reason. Since  $\xi|_{R_\xi} = 0$  (as a Čech cohomology class), for any smooth closed 1-form  $\omega$  on  $M$  representing  $\xi$  there exists a smooth function  $f$  defined on an open neighborhood of  $R_\xi$  such that  $\omega = df$  near  $R_\xi$ . Then we obtain

$$(8.5) \quad \langle \xi, \mathcal{A}_{\mu_1} \rangle = \int_M \iota_V(\omega) d\mu_1 = \int_{R_\xi} \iota_V(\omega) d\mu_1 = \int_{R_\xi} V(f) d\mu_1 = 0.$$

The last equality holds since the measure  $\mu_1$  is  $\Phi$ -invariant (see e.g. [16], Theorem on page 277). Finally, as  $\mathcal{A}_\mu = \mathcal{A}_{\mu_1} + \mathcal{A}_{\mu_2}$  we see that  $\langle \xi, \mathcal{A}_\mu \rangle \leq -\eta \cdot \mu(C_\xi)$  with  $\eta = \alpha/2$ , which completes the proof of (II)  $\Rightarrow$  (III).

The implication (III)  $\Rightarrow$  (IV) is obvious.

We are left to show the implication (IV)  $\Rightarrow$  (I). Our argument uses the technique of Schwartzman [16]. It is to show that under the conditions (IV) there exists a smooth Lyapunov 1-form for  $(\Phi, R_\xi)$  in the class  $\xi$ . In a first step we prove that there exists a smooth, closed 1-form  $\omega_1$  in the class  $\xi$  so that  $\iota_V(\omega_1) < 0$  on  $C_\xi$ . To this end, denote by  $\mathcal{D} \subset C^0(M)$  the space of functions

$$\mathcal{D} = \{V(f); f: M \rightarrow \mathbf{R} \text{ is smooth}\}$$

and by  $\mathcal{C}^-$  the convex cone in  $C^0(M)$  consisting of all functions  $f \in C^0(M)$  with

$$f(x) < 0 \quad \text{for all } x \in C_\xi.$$

As  $C_\xi$  is compact, the cone  $\mathcal{C}^-$  is open in the Banach space  $C^0(M)$  of continuous functions on  $M$ , endowed with the usual supremum norm. Choose an arbitrary smooth, closed 1-form  $\omega$  in the class  $\xi$ . Assume that  $\mathcal{C}^- \cap (\iota_V(\omega) + \mathcal{D}) = \emptyset$ . It then follows from the Hahn–Banach Theorem (cf. [15], p. 58) that there exists a continuous linear functional  $\Lambda: C^0(M) \rightarrow \mathbf{R}$  such that

$$\Lambda|_{\iota_V(\omega) + \mathcal{D}} \geq 0 \quad \text{and} \quad \Lambda|_{\mathcal{C}^-} < 0.$$

Since  $\iota_V(\omega) + \mathcal{D}$  is an affine subspace and  $\Lambda$  is bounded on it from below, we obtain that  $\Lambda$  restricted to  $\mathcal{D}$  vanishes. According to the Riesz representation theorem (cf. [12]), there exists a Borel measure  $\mu$  on  $M$  so that

$$\Lambda(f) = \int_M f d\mu$$

for any  $f \in C^0(M)$ . By Theorem [16], p. 277, the condition  $\Lambda|_{\mathcal{D}} = 0$  implies that  $\mu$  is  $\Phi$ -invariant. On the other hand,  $\Lambda|_{\mathcal{C}^-} < 0$  implies that  $\mu|_{C_\xi} > 0$ .

Denote by  $\chi: M \rightarrow \mathbf{R}$  the characteristic function of  $C_\xi$  and let  $\nu = \chi \cdot \mu$ . As  $C_\xi$  is  $\Phi$ -invariant  $\nu$  is a  $\Phi$ -invariant Borel measure and (unlike, possibly,  $\mu$ ) is positive. Note that  $\mu - \nu$  is a  $\Phi$ -invariant Borel measure supported on  $R_\xi$  (again using that any  $\Phi$ -invariant measure is supported on  $R = R_\xi \cup C_\xi$ ). Thus, it follows from our assumption  $\xi|_{R_\xi} = 0$ , by the same argument which led to (8.5), that

$$\langle \xi, \mathcal{A}_{\mu-\nu} \rangle = 0.$$

Since  $\mathcal{A}_{\mu-\nu} = \mathcal{A}_\mu - \mathcal{A}_\nu$  we find

$$\langle \xi, \mathcal{A}_\nu \rangle = \langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega) d\mu = \Lambda(f) \geq 0$$

where  $f = \iota_V(\omega)$ , contradicting condition (6.2). This means that the intersection  $\mathcal{C}^- \cap (\iota_V(\omega) + \mathcal{D})$  cannot be empty, i.e. there exists a smooth function  $g: M \rightarrow \mathbf{R}$  so that the smooth closed 1-form  $\omega_1 = \omega + dg$  is in the class  $\xi$  and satisfies

$$\iota_V(\omega_1) < 0 \quad \text{on } C_\xi.$$

This completes the first step of the proof.

To finish the argument, we now adjust  $\omega_1$  on the complement of  $C_\xi$  so that the resulting form is a Lyapunov 1-form for  $(\Phi, R_\xi)$ . As  $\iota_V(\omega_1) < 0$  on  $C_\xi$  and  $C_\xi$  is compact, there is some open neighborhood  $W_1$  of  $C_\xi$  such that  $W_1 \cap R_\xi = \emptyset$  and  $\iota_V(\omega_1) < 0$  on  $W_1$ . Since  $\xi|_{R_\xi} = 0$ , there exists an open neighborhood  $W_2$  of  $R_\xi$  such that  $W_1 \cap W_2 = \emptyset$  and a smooth function  $g: M \rightarrow \mathbf{R}$  such that  $\omega_1|_{W_2} = dg$  and  $dg|_{W_1} = 0$ . By Proposition 2 there exists a smooth Lyapunov function  $L: M \rightarrow \mathbf{R}$  for  $(\Phi, R)$ . Now consider

$$(8.6) \quad \omega_2 = \omega_1 - dg + \lambda dL,$$

where  $\lambda > 0$  remains to be chosen. Clearly, the form  $\omega_2$  is smooth and closed and represents the class  $\xi$ . For any  $\lambda > 0$  it satisfies  $\omega_2|_{W_2} = d(\lambda L)$ , because  $\omega_1 - dg$  vanishes on this set. In particular,  $\omega_2$  has property (A2) of a Lyapunov 1-form for the pair  $(\Phi, R_\xi)$ . Note also that for all positive  $\lambda$  we have  $\iota_V(\omega_2) < 0$  on  $W_1$  (by the construction of  $W_1$ ) and on  $W_2 - R_\xi$  because  $\omega_1 - dg$  vanishes there, whereas  $V(L) < 0$ . As the complement of  $W_1 \cup W_2$  is compact and disjoint from  $R$ ,

$$1 < \lambda_0 := 1 + \sup_{x \notin W_1 \cup W_2} \frac{|\iota_V(\omega_1 - dg)|}{|V(L)|} < \infty,$$

and  $\iota_V(\omega_2) < 0$  on  $M - R_\xi$  for all  $\lambda \geq \lambda_0$ , showing that for such choices of  $\lambda$  the form  $\omega_2$  also has property (A1) of a Lyapunov 1-form for  $(\Phi, R_\xi)$ . This completes the proof of the implication (IV)  $\Rightarrow$  (I) and hence the proof of Theorem 1.  $\square$

#### APPENDIX: PROOF OF PROPOSITION 2

Recall from [1, II 6.2.A] the alternative characterization of the chain recurrent set  $R$  as

$$R = \bigcap \{A \cup A^* \mid (A, A^*) \text{ is an attractor-repeller pair}\}.$$

Here a closed, flow-invariant subset  $A \subset M$  is called an attractor if it admits a neighborhood  $U$  such that  $A$  is the maximal flow-invariant subset in the closure of  $U \cdot [0, \infty)$ . The dual repeller  $A^*$  is the set of all points  $x \in M$  whose forward limit set is disjoint from  $A$  (cf. [1, II 5.1]). Equivalently,  $(A, A^*)$  is an attractor-repeller pair if and only if both  $A$  and  $A^*$  are closed flow invariant subsets of  $M$  and the forward (resp. backward) limit set of every point  $x \notin A \cup A^*$  is contained in  $A$  (resp.  $A^*$ ) – see [14, Prop. 1.4].

As  $M$  is a closed manifold and hence separable, the number of distinct attractor-repeller pairs is at most countable (cf. [1, II 6.4.A]). Let  $\{(A_n, A_n^*)\}_{n \geq 1}$  be some enumeration. For each  $n \geq 1$ , the construction of Robbin and Salamon (Prop. 1.4. of [14] and the remark following it) yields a smooth function  $f_n: M \rightarrow [0, 1]$  with  $f_n^{-1}(0) = A_n$ ,  $f_n^{-1}(1) = A_n^*$  and  $df_n(V) < 0$  on the complement of  $A_n \cup A_n^*$ . Let  $c_n$  be positive constants such that in a fixed finite atlas of charts all partial derivatives of  $f_n$  of order  $\leq n$  are bounded pointwise in absolute value by  $c_n$ . Then

$$L(x) := \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n c_n}$$

is a smooth function having the required properties. In particular, as for any  $n \geq 1$  the differential  $df_n$  vanishes on  $A_n \cup A_n^*$ , the differential of  $L$  vanishes on  $R$ .  $\square$

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