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CANTORIAN TABLEAUX AND PERMANENTS

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ABSTRACT. This article could be called "theme and variations" on Cantor's celebrated diagonal argument. Given a square $n \times n$ tableau $T = (a_i^j)$ on a finite alphabet A, let L be the set of its row-words. The permanent $\mathrm{Perm}(T)$ is the set of words $a^1_{\pi(1)}a^2_{\pi(2)} \cdots a^n_{\pi(n)}$, where π runs through the set of permutations of n elements. Cantorian tableaux are those for which $\mathrm{Perm}(T) \cap L = \varnothing$. Let s be the cardinality of A. We show in particular that for large n, if $s < (1-\epsilon)n/\log n$ then most of the tableaux are non-Cantorian, whereas if $s > (1+\epsilon)n/\log n$ then most of the tableaux are Cantorian. We conclude our article by the study of infinite tableaux. Consider for example the infinite tableau whose rows are the binary expansions of the real algebraic numbers in the unit interval. We show that the permanent of this tableau contains exactly the set of binary expansions of *all* the transcendental numbers in the unit interval.

1. INTRODUCTION

We all know Cantor's proof of the existence of transcendental numbers, namely writing the expansions in a base $s > 1$ of the algebraic numbers in the interval $(0, 1)$ into a tableau and considering its diagonal. This diagonal was used by Cantor in order to construct the expansion of a transcendental number. We show in Theorem 10 that the diagonal of such a tableau is itself the expansion of a transcendental number and has some additional properties. Furthermore we show that by permuting the rows of the infinite tableau we obtain essentially all the transcendental numbers. However, before this we consider the natural analogue for finite tableaux. We call these Cantorian if neither their diagonal nor that of any row permutation is equal to a row.

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2. DEFINITIONS

Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$, $s \geq 2$ be a finite alphabet and let T be a square $n \times n$ tableau

$$T = \begin{pmatrix} a_1^1 & a_1^2 & \ldots & a_1^n \\ a_2^1 & a_2^2 & \ldots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \ldots & a_n^n \end{pmatrix}, \quad a_i^j \in A .$$

Each row $l_i = a_i^1 a_i^2 \cdots a_i^n$ is considered as a word of length n. The sequence of rows is denoted by $\bar{L}$ and the set of distinct row-words is denoted by L. It contains at most n words.

The permanent of an $n \times n$ matrix (a_i^j) defined on a ring is

$$\sum_{\pi \in S_n} a_{\pi(1)}^1 a_{\pi(2)}^2 \cdots a_{\pi(n)}^n ,$$

where the summation is over the set of permutations of the n elements. Very naturally we define the permanent of the tableau T to be the set of words

$$\mathrm{Perm}(T) = \bigcup_{\pi \in S_n} a_{\pi(1)}^1 a_{\pi(2)}^2 \cdots a_{\pi(n)}^n .$$

This set contains in particular the diagonal word

$$\mathrm{Diag}(T) = a_1^1 a_2^2 \cdots a_n^n .$$

Note that the permanents of two tableaux that differ only in the order of their rows are the same. It may be useful to note that

$$\mathrm{Perm}(T) = \{\mathrm{Diag}(T') \mid T' \text{ a tableau obtained from } T \text{ by permuting its rows}\}.$$

Cantor's famous diagonal argument is based on the comparison of the set of rows of an infinite tableau with its diagonal [3]. Here, at least in the beginning of the present article, we are mostly concerned with finite $n \times n$ tableaux and their diagonals. The last section is dedicated to infinite tableaux.

DEFINITION. A tableau is *Cantorian* if none of its row-words appear in $\mathrm{Perm}(T)$. In symbols

$$L \cap \mathrm{Perm}(T) = \varnothing .$$

Here are some examples on the two letter alphabet $A = \{a, b\}$:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} a & a & b & a & a & b \\ b & b & a & b & b & a \\ a & b & a & b & a & b \\ b & a & b & a & b & a \\ b & b & b & a & b & b \\ a & a & a & b & a & a \end{pmatrix}, \quad \begin{pmatrix} a & b & a \\ b & a & b \\ b & b & b \end{pmatrix}.$$

The first one is clearly Cantorian. Verifying that the second one also is Cantorian seems to be a formidable task, since $\mathrm{Perm}(T)$ consists of $6! = 720$ words (not all distinct since $2^6 < 6!$). In Section 3 we present a simple condition which establishes that the tableau is Cantorian. On the other hand, the third one is not since $bbb \in L \cap \mathrm{Perm}(T)$.

REMARK. Note that the property of being Cantorian is invariant under permutation of rows and permutation of columns, and, given any bijection on the alphabet, replacing all entries of a column by their image under this bijection. To illustrate the latter, consider the following two tableaux:

$$\begin{pmatrix} a & a & b & b & c \\ a & a & b & b & c \\ a & a & b & b & c \\ b & b & a & a & d \\ b & b & a & a & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & a & a & a & a \\ a & a & a & a & a \\ a & a & a & a & a \\ b & b & b & b & b \\ b & b & b & b & b \end{pmatrix}.$$

While it might be difficult to see whether the first of them is Cantorian or not, it is clear that the second is in fact Cantorian. However, it differs from the first one only by exchanging a's and b's in column three and four, and writing a instead of c and b instead of d in column five. Hence, both of the two tableaux must be Cantorian.

How can we calculate a permanent ? There is actually an induction formula which is similar to the one for determinants. Given an $n \times n$ tableau T, let T_i^j be the $(n-1) \times (n-1)$ tableau obtained by deleting row i and column j. Let

$$\mathrm{Ins}^{\to j}\left(\alpha, \mathrm{Perm}(T_i^j)\right)$$

denote the set of words obtained by inserting the letter α at the j-th place of each word in $\mathrm{Perm}(T_i^j)$.

THEOREM 1. *For all* $i \in \{1, 2, \ldots, n\}$

$$\mathrm{Perm}(T) = \bigcup_{j=1}^{n} \mathrm{Ins}^{\rightarrow j}\left(a_i^j, \mathrm{Perm}(T_i^j)\right).$$

Proof. The proof is obvious. □

COROLLARY 2. *Let* T *be an* $n \times n$ *tableau over the alphabet* A. *Suppose a letter – say* a *– occurs* $n^2 - n + 1$ *times or more often in* T. *Then* T *is non-Cantorian. More specifically*

$$a^n \in L \cap \mathrm{Perm}(T).$$

If a *occurs only* $n^2 - n$ *times, the result need not be true.*

Proof. If T contains no letter other than a the result is trivial. If not, we argue by induction on n. If $n = 1$ the result is trivially true. Otherwise let T be an $n \times n$ tableau which contains at least $n^2 - n + 1$ occurrences of the letter a. There is at least one row, say the i-th, which contains no letter other than a. Since T contains at least one letter different from a, there is a column j with at most $n - 1$ occurrences of a. Thus T_i^j contains at least $n^2 - n + 1 - n - (n-2) = (n-1)^2 - (n-1) + 1$ occurrences of a. By hypothesis

$$a^{n-1} \in \mathrm{Perm}(T_i^j)$$

and therefore

$$a^n \in \mathrm{Ins}^{\rightarrow j}\left(a, \mathrm{Perm}(T_i^j)\right).$$

By Theorem 1 we have that $a^n \in \mathrm{Perm}(T)$.

On the other hand, it is easy to see that the following $n \times n$ tableau with $n^2 - n$ occurrences of a is Cantorian:

$$\begin{pmatrix} a & a & \ldots & a \\ a & a & \ldots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \ldots & a \\ b & b & \ldots & b \end{pmatrix}. \tag{1}$$

□

The problem remains to characterize $n \times n$ Cantorian tableaux. We shall not be able to give a definite answer to this question. We shall however provide a sufficient condition which implies that our first two examples are indeed Cantorian.

3. A SUFFICIENT CONDITION

Let Σ be the set of all maps $\sigma: A \to A$ with no fixed points: $\forall \alpha \in A : \sigma(\alpha) \neq \alpha$. If $\alpha\alpha'\alpha'' \cdots$ is a finite or infinite word on A, we define

$$\sigma(\alpha\alpha'\alpha'' \cdots) = \sigma(\alpha)\sigma(\alpha')\sigma(\alpha'') \cdots .$$

Let $\overline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma^n$. Recall that $\bar{L} = (l_1, l_2, \ldots, l_n)$ is the sequence of row-words of the tableau T. Define

$$\overline{\sigma}\bar{L} = (\sigma_1 l_1, \sigma_2 l_2, \ldots, \sigma_n l_n)$$

and let $\overline{\sigma}L$ denote the set of all distinct $\sigma_i l_i$. Finally, let $\overline{\sigma}T$ be the $n \times n$ tableau whose row-words are $\sigma_1 l_1, \sigma_2 l_2, \ldots, \sigma_n l_n$, that is

$$\overline{\sigma}T = \begin{pmatrix} \sigma_1 a_1^1 & \sigma_1 a_1^2 & \ldots & \sigma_1 a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n a_n^1 & \sigma_n a_n^2 & \ldots & \sigma_n a_n^n \end{pmatrix} .$$

THEOREM 3. *Let* $\overline{\sigma} \in \Sigma^n$. *Then*

$$\operatorname{Perm}(T) \cap \overline{\sigma}L = \varnothing \tag{2}$$

and

$$\operatorname{Perm}(\overline{\sigma}T) \cap L = \varnothing . \tag{3}$$

Proof. (2) Suppose $\operatorname{Perm}(T) \cap \overline{\sigma}L \neq \varnothing$. Thus there is a permutation $\pi \in S_n$ and an index $i \in \{1, 2, \ldots, n\}$ such that

$$a^1_{\pi(1)} a^2_{\pi(2)} \cdots a^n_{\pi(n)} = \sigma_i(l_i) = \sigma_i(a_i^1)\sigma_i(a_i^2) \cdots \sigma_i(a_i^n) .$$

Let j be $\pi^{-1}i$. Comparing the j-th letter on both sides we obtain $a_i^j = \sigma_i(a_i^j)$, which contradicts $\sigma_i \in \Sigma$.

(3) Suppose now $\operatorname{Perm}(\overline{\sigma}T) \cap L \neq \varnothing$. There is a permutation π and an index i such that

$$\sigma_{\pi(1)}(a^1_{\pi(1)})\sigma_{\pi(2)}(a^2_{\pi(2)}) \cdots \sigma_{\pi(n)}(a^n_{\pi(n)}) = a_i^1 a_i^2 \cdots a_i^n .$$

As above let $j = \pi^{-1}(i)$ and consider the j-th letter on both sides. We obtain

$$\sigma_i(a_i^j) = a_i^j ,$$

which again contradicts $\sigma_i \in \Sigma$. □

COROLLARY 4. *If $L = \overline{\sigma}L$, then the tableau is Cantorian.*

It is easy to see that Theorem 3 holds in fact – mutatis mutandis – for fixed-point free relations σ, i.e. relations with $\alpha \notin \sigma(\alpha)$ for $\alpha \in A$. As a corollary we obtain

COROLLARY 5. *If for every row i there exists a row i' such that $a_i^j \neq a_{i'}^j$ for all j, then the tableau is Cantorian.*

Note that the only admissible map $\sigma \in \Sigma$ on a two letter alphabet $\{a, b\}$ is $\sigma(a) = b$, $\sigma(b) = a$. Thus, in this case the Corollaries 4 and 5 state the same thing.

All our previous examples of Cantorian tableaux are of the type described by the corollary above. There are, however, other Cantorian tableaux, as the following examples show:

$$\begin{pmatrix} a & a & a & a \\ a & a & a & a \\ b & b & b & a \\ b & b & b & b \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} a & a & a & a & a \\ a & a & a & a & a \\ b & b & b & a & b \\ b & b & b & b & a \\ b & b & b & b & b \end{pmatrix}.$$

The following fixed point theorem is another consequence of Theorem 3. Before stating it we need to extend the concept of a permanent. Let $W = \{w_1, w_2, \ldots, w_m\}$ be a set of m distinct words of length $n \geq m$. If $m < n$, repeating some of these words, we can obtain an $n \times n$ tableau T. Ignoring permutations, there are actually $\binom{n-1}{m-1}$ ways to construct such a tableau containing all the words of W. To each one of these T corresponds a permanent Perm(T). We define Perm(W) as the union of all the $\binom{n-1}{m-1}$ permanents.

COROLLARY 6. *Let $W = \{w_1, w_2, \ldots, w_m\}$ be a set of $m \leq n$ words, each of length n. Suppose that $\sigma: A \to A$ is a map such that $\sigma(W) \subset W$ and $W \cap \mathrm{Perm}(W) \neq \varnothing$. Then there exists a letter $\alpha \in A$ such that $\sigma(\alpha) = \alpha$.*

Proof. Obvious from Theorem 3 with $\overline{\sigma} = (\sigma, \sigma, \ldots, \sigma)$. □

4. Counting Cantorian tableaux

Let us denote by $c(n,p)$ the number of Cantorian tableaux of size $n \times n$ over the alphabet $\{a,b\}$ having exactly p occurrences of b. Clearly $c(n,p)$ has a symmetric distribution with respect to p, that is to say, $c(n,p) = c(n,n^2-p)$. We also have the following computational evidence:

TABLE 1
$c(n,p)$

$n\backslash p$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	4	0	0										
3	0	0	3	9	9	3	0	0	0					
4	0	0	0	4	0	112	384	744	384	112	0	4	0	...
5	0	0	0	0	5	0	0	275	1650	5960	14250	22100	22100	...

These numbers suggest the following result:

THEOREM 7. *Let $c(n,p)$ be the number of Cantorian tableaux over the alphabet $\{a,b\}$ with exactly p occurrences of the letter b. We have*

$$c(n,p) = \begin{cases} 0 & \text{for } p < n, \\ n & \text{for } p = n \geq 3, \\ 0 & \text{for } p = n+1 \text{ and } n \geq 4, \\ 0 & \text{for } p = n+2 \text{ and } n \geq 5. \end{cases}$$

REMARK. For $p = n+3$ and $n \geq 3$, the following tableau is Cantorian:

$$\begin{pmatrix} & & & & & \\ & & & & & \\ & & & & b & b & b \\ b & \dots & b & b & b & b \end{pmatrix},$$

where all the entries which are not indicated are a's. Hence, $c(n,n+3)$ does not vanish.

Proof. The n Cantorian tableaux for $p = n$ are those obtained by permuting the rows of the tableau displayed in (1) in the proof of Corollary 2. We will show that there are no others by considering several cases.

• If $p < n$, Corollary 2 applies and we are done.

• If $n \leq p \leq n+2$ and if there is a row b^n, direct inspection of the possible cases shows that the tableau is Cantorian if and only if $p = n$. Up to permutation of rows and columns, the possible cases are

$$\begin{pmatrix} & & \\ b & \dots & b \; b \end{pmatrix}, \quad \begin{pmatrix} & & & b \\ b & \dots & b & b \end{pmatrix},$$

$$\begin{pmatrix} & & b & b \\ b & \dots & b & b \end{pmatrix}, \quad \begin{pmatrix} & & & b \\ & & & b \\ b & \dots & b & b \end{pmatrix}, \quad \begin{pmatrix} & & b & \\ & & & b \\ b & \dots & b & b \end{pmatrix},$$

where all the entries which are not indicated are a's.

• If $n \leq p \leq n+2$ and there is no row b^n, but a row a^n we claim that a^n is in the permanent of T: let i be the row with the greatest number of b's. Because there is a row without any b's and the total number of b's is at least n, there are at least two letters b in this row. Since there is no row b^n, this row contains at least one letter a, say in column j.

Now we consider the tableau T_i^j. Clearly, a^{n-1} is a row of T_i^j. Furthermore, note that b^{n-1} cannot be a row of T_i^j: this would be possible only if there were two rows b^{n-1} in T. Since the number of b's in T is p, we would have $p \geq 2(n-1)$, and hence $n \leq p - n + 2$, which contradicts the bounds we assumed for n.

We proceed by induction on n: since T_i^j contains at most $p-2$ letters b, we have by hypothesis that a^{n-1} is in the permanent of T_i^j. Since $\mathrm{Ins}^{\to j}\left(a, \mathrm{Perm}(T_i^j)\right)$ is a subset of the permanent of T, we are done.

• If $n \leq p \leq n+2$ and there is neither a row b^n, nor a row a^n, we have to consider two subcases:

— There is a column j with at least two b's, with one of them being the only one in its row. Let i be one of the other rows having a b in column j. Consider the tableau T_i^j. Clearly, it contains a row a^{n-1}. Because T has no row a^n, the reduced tableau T_i^j cannot contain a row b^{n-1}: otherwise we would have $p \geq n - 1 + 2 + n - 3 = 2n - 2$, which contradicts the bounds we assumed for n.

Now the statement established in the previous case applies to T_i^j, and by Theorem 1 we obtain that row i is in the permanent of T, which contains $\mathrm{Ins}^{\to j}\left(b, \mathrm{Perm}(T_i^j)\right)$.

— Otherwise, by permuting rows and columns, the tableau can be represented as

$$\begin{pmatrix} b & & & & \\ & \ddots & & & \\ & & b & & \\ & & & b & b \\ & & & b & b \end{pmatrix}, \tag{4}$$

where all the entries which are not indicated are a's. Clearly, this tableau is non-Cantorian. □

Let $C(n,s)$ be the number of $n \times n$ Cantorian tableaux on an s-letter alphabet. Computing $C(n,s)$ is obviously quite cumbersome even for $s = 2$. As a first improvement over simple-minded calculation of the permanent followed by checking whether the intersection with the set of row-words is nonempty, we have the following:

THEOREM 8. *It is possible to test whether a given tableau is Cantorian or not in polynomial time.*

Proof. To test whether a given $n \times n$ tableau T over any alphabet is Cantorian or not, we proceed as follows: for each row k we transform the tableau into a bipartite graph G_k, with each of the two parts having n vertices. The n 'top' vertices $\{x_1, x_2, \ldots, x_n\}$ correspond to the rows of the tableau, the n 'bottom' vertices $\{y_1, y_2, \ldots, y_n\}$ correspond to the columns of T. There is an edge connecting the 'top' vertex x_i with the 'bottom' vertex y_j if and only if the entries in column j in row i and in row k are the same, i.e., if $a_i^j = a_k^j$.

If G_k contains a perfect matching, then the tableau T cannot be Cantorian. Otherwise, we proceed with the next line of the tableau. If there is no perfect matching for any of the rows of the tableau, it must be Cantorian.

Since it is possible to find a perfect matching in a given bipartite graph in time $O(n^{2.5}/\sqrt{\log n})$, where n is the number of vertices of the graph – see [2, Theorem 1] –, we see that this procedure is polynomial in time. □

For very small n, we can inspect all $n \times n$ tableaux and check whether they are Cantorian. Since, even for a two letter alphabet, the number of tableaux is 2^{n^2}, this rapidly becomes unfeasible as a method for determining their number. We have computed the number of Cantorian tableaux of sizes up to 11 over

a two letter alphabet by an algorithm which uses the three Cantor-preserving operations on tableaux (described in the Remark of Section 2) to divide the Cantorian tableaux into a relatively manageable number of classes each of easily computed cardinality. A forthcoming paper by the last two authors will describe this algorithm in detail. The computation for size 11 took about a week. The results can be found in Table 2.

5. ASYMPTOTICS

Consider the 16 tableaux of size 2×2 over the alphabet $\{a, b\}$. Direct inspection shows that among them, only 4 are Cantorian. There are 2^{n^2} tableaux of size $n \times n$. It is reasonable to guess that among them there is only a small proportion of Cantorian tableaux. Let $C(n) = C(n, 2)$ be the number of Cantorian tableaux and $N(n)$ the number of non-Cantorian tableaux over $\{a, b\}$. We have the following explicit bounds:

$$N(n) \geq 2^{n^2-n+1} \quad \text{and} \quad C(n) \geq \begin{cases} 2^{\frac{1}{2}n^2} & \text{if } n \text{ is even,} \\ 2^{\frac{1}{2}n(n-1)} & \text{if } n \text{ is odd.} \end{cases}$$

In particular, the lower bound for $C(n)$ is obtained as follows. Suppose n is even. Choose arbitrarily the entries in the first $n/2$ rows of an $n \times n$ tableau T. Complete T by adjoining $n/2$ rows obtained by interchanging the letters a and b in the first $n/2$ rows of T. By Corollary 4 this tableau is necessarily Cantorian, and there are $2^{\frac{1}{2}n^2}$ such tableaux. Obvious modifications establish the case where n is odd.

Therefore

$$\frac{1}{2} \leq \liminf_{n\to\infty} \frac{\log C(n)}{\log 2^{n^2}} \leq \limsup_{n\to\infty} \frac{\log C(n)}{\log 2^{n^2}} \leq 1 .$$

The question arises whether the limit actually exists, and if so, what is its value? Here are a few values of the ratio

$$\frac{\log C(n)}{\log 2^{n^2}} = .5, .509, .673, .657, .675, .656, .651, .632, .626, .610$$

for respectively $n = 2, 3, \ldots, 11$, computed from Table 2 which displays the values of the number $C(n, s)$ of $n \times n$ Cantorian tableaux on an alphabet of size s, for some values of n and s.

TABLE 2
Number and proportion of Cantorian tableaux of size n on alphabets of size s

$n\backslash s$	2	3	4	5	6
2	$1 \cdot 2^2$ $2.5 \cdot 10^{-1}$	$4 \cdot 3^2$ $\sim 4.44 \cdot 10^{-1}$	$9 \cdot 4^2$ $5.625 \cdot 10^{-1}$	$16 \cdot 5^2$ $6.4 \cdot 10^{-1}$	$25 \cdot 6^2$ $\sim 6.94 \cdot 10^{-1}$
3	$3 \cdot 2^3$ $\sim 4.69 \cdot 10^{-2}$	$188 \cdot 3^3$ $\sim 2.58 \cdot 10^{-1}$	$1863 \cdot 4^3$ $\sim 4.55 \cdot 10^{-1}$	$9264 \cdot 5^3$ $\sim 5.93 \cdot 10^{-1}$	$32075 \cdot 6^3$ $\sim 6.87 \cdot 10^{-1}$
4	$109 \cdot 2^4$ $\sim 2.66 \cdot 10^{-2}$	$100144 \cdot 3^4$ $\sim 1.88 \cdot 10^{-1}$			
5	$2765 \cdot 2^5$ $\sim 2.64 \cdot 10^{-3}$				
6	$324781 \cdot 2^6$ $\sim 3.02 \cdot 10^{-4}$				
7	$37304106 \cdot 2^7$ $\sim 8.48 \cdot 10^{-6}$				
8	$13896810621 \cdot 2^8$ $\sim 1.93 \cdot 10^{-7}$				
9	$5438767247337 \cdot 2^9$ $\sim 1.15 \cdot 10^{-9}$				
10	$6889643951630251 \cdot 2^{10}$ $\sim 5.56 \cdot 10^{-12}$				
11	$8135113082369752094 \cdot 2^{11}$ $\sim 6.27 \cdot 10^{-15}$				

The proportion of Cantorian tableaux on a fixed s-letter alphabet tends to 0 as the size of the tableaux increases, as is suggested by Table 2 and established by the following theorem:

THEOREM 9. *Let $C(n, s)$ be the number of $n \times n$ Cantorian tableaux on an alphabet of size $s = s(n)$. If $s < n/(\log n + \log\log n + r_n)$ where r_n is any sequence which grows without bound, then*

$$\lim_{n\to\infty} C(n, s)/s^{n^2} = 0 .$$

If on the other hand, $s > n/(\log n - \log\log n - \epsilon)$ for any $\epsilon > 0$, then

$$\lim_{n\to\infty} C(n, s)/s^{n^2} = 1 .$$

Proof. A tableau T is certainly non-Cantorian if $l_n \in \mathrm{Perm}(T)$. This in turn is certainly the case if there is a permutation $\pi \in S_{n-1}$ such that

$a_i^{\pi(i)} = a_n^{\pi(i)}$ for all $i < n$. If π consists of a single cycle, then the following directed graph G has a (directed) Hamiltonian cycle: $G = (V, E)$, where $V = \{1, 2, \ldots, n-1\}$ and $(i,j) \in E$ if and only if $a_i^j = a_n^j$. Note that G is derived from the graph G_n in Section 4 by omitting vertices x_n and y_n, directing all its edges from 'top' to 'bottom' and then identifying vertices x_i and y_i for $i \in \{1, 2, \ldots, n-1\}$.

This graph G for a randomly chosen $n \times n$ tableau is simply $D_{n-1,1/s}$, a random directed graph on $n-1$ vertices where each possible edge has probability $1/s$ of occurring. The probability that such a graph is Hamiltonian is well known to tend to 1 as n tends to infinity [6], as long as the alphabet size $s(n)$ grows with n but is bounded by $s(n) < n/(\log n + \log\log n + r_n)$ where r_n is any sequence which grows without bound.

On the other hand, if $s(n) > n/(\log n - \log\log n - \epsilon)$ for $\epsilon > 0$, then Corollary 5 shows that the probability that a random tableau is Cantorian tends to 1. Indeed, for any two rows i and j $(i \neq j)$,

$$\begin{aligned}\Pr[\forall k : a_i^k \neq a_j^k] &> (1 - (\log n - \log\log n - \epsilon)/n)^n \\ &> \log n(1 + \epsilon')/n \qquad (\epsilon' > 0)\end{aligned}$$

for n sufficiently large. Hence for a given i,

$$\begin{aligned}\Pr[\exists j \forall k : a_i^k \neq a_j^k] &> 1 - (1 - \log n(1 + \epsilon')/n))^{n-1} \\ &> 1 - n^{-(1+\epsilon'')} \qquad (\epsilon'' > 0)\end{aligned}$$

for n sufficiently large. We deduce that the expected number of i such that $\nexists j \forall k : a_i^k \neq a_j^k$ is less than $n^{-\epsilon''}$. Therefore the probability that there exists such an i is less than $n^{-\epsilon''}$ and thus tends to 0. □

6. INFINITE TABLEAUX

The definitions and results of the preceding sections extend naturally to infinite tableaux $T = (a_i^j)$ with $i, j \in \mathbf{N}$. In particular the permanent of T is the set of infinite sequences

$$\mathrm{Perm}(T) = \bigcup_{\pi \in S_{\mathbf{N}}} a_{\pi(1)}^1 a_{\pi(2)}^2 a_{\pi(3)}^3 \cdots,$$

where $S_{\mathbf{N}}$ is the family of all bijections $\pi : \mathbf{N} \to \mathbf{N}$. In general, $\mathrm{Perm}(T)$ is an uncountable set.

Consider an infinite tableau T over the alphabet A. The i-th row ($i \in \mathbf{N}$) is

$$l_i = a_i^1 a_i^2 a_i^3 \cdots \in A^{\mathbf{N}}$$

and the set of rows is denoted L as in the finite case. If $\overline{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \dots) \in \Sigma^{\mathbf{N}}$, where Σ is the set of maps $A \to A$ without fixpoints (see Section 3), then Theorem 3 extended to the infinite case asserts that

$$\mathrm{Perm}(T) \cap \overline{\sigma} L = \varnothing \text{ and } \mathrm{Perm}(\overline{\sigma} T) \cap L = \varnothing .$$

If moreover $L = \overline{\sigma} L$, then T is Cantorian.

THEOREM 10. *Let L be a countable subset of the unit interval such that $L \supseteq \mathbf{Q} \cap [0,1]$. Let T be the infinite tableau whose rows are the expansions in base $s \geq 2$ of the numbers in L. Here we require that rational numbers r/s^q for $r, q \in \mathbf{N}$ should appear twice in T, once with a tail of 0's and once with a tail of $s-1$'s. Suppose that, if $s > 2$, $L + 1/(s-1) \equiv L \bmod 1$ and if $s = 2$, $L = 1 - L$. Then T is Cantorian:* $\mathrm{Perm}(T)$ *only contains numbers in* $[0,1] \setminus L$. *Furthermore the s-expansion of each number in* $\mathrm{Perm}(T)$ *contains each of the digits $0, 1, \dots, s-1$ infinitely often. None of the digits occur periodically.*

REMARK. In fact, for $s \geq 3$ the theorem holds also if the tableau contains only one of the two possible expansions of rational numbers r/s^q for $r, q \in \mathbf{N}$, i.e., the expansion having an infinite tail of 0's.

Note that this does not work for $s = 2$. Rather, in this case the intersection $\mathrm{Perm}(T) \cap L$ contains exactly the numbers whose expansions have an infinite tail of 0's. The statement that both digits 0 and 1 occur infinitely often for numbers in $\mathrm{Perm}(T)$ is false in this setting.

Proof. Let $l_i = a_i^1 a_i^2 a_i^3 \cdots$ represent both the i-th row of T and the i-th element of L. Assume that $s \geq 3$. In this case, we define

$$\pi : l_i \mapsto l_i + 1/(s-1) \bmod 1 .$$

Clearly, π is a reordering of the rows. Moreover, since $1/(s-1) = .1111\dots$, it can be shown that the j-th digit of $\pi(l_i)$ is always different from the j-th digit of l_i. For $s = 2$ we define $\pi : l_i \mapsto 1 - l_i$, which, taking into account that a rational number with terminating expansion appears also with a tail of ones, implies the same fact.

By Corollary 5 we conclude that the tableau is Cantorian, i.e., Perm(T) contains only expansions of numbers in $[0,1] \setminus L$.

Next we prove that each of the s-digits $0, 1, \ldots, s-1$ occurs infinitely often in every element of Perm(T). Suppose the digit a occurs only finitely many times in

$$p = a^1_{\pi(1)} a^2_{\pi(2)} \cdots .$$

Define the map $\sigma\colon \{0,1,\ldots,s-1\} \to \{0,1,\ldots,s-1\}$ by $\sigma(b) = a$ for all $b \neq a$ and $\sigma(a) = c$, where c is any letter different from a. Then $\sigma \in \Sigma$ and therefore Perm($\overline{\sigma}T$) contains no numbers from L. Thus

$$\sigma(p) = \sigma(a^1_{\pi(1)})\sigma(a^2_{\pi(2)}) \cdots$$

is the s-expansion of a number in $[0,1] \setminus L$. But from some point on, p contains no digit a, so that $\sigma(p)$ has an infinite tail of a's. This is absurd since then $\sigma(p) \in \mathbf{Q}$. The same map σ shows that no s-digit of p can occur periodically. □

REMARK. Theorem 9 asserts that given an alphabet A, with a fixed number s of elements, the probability that an $n \times n$ tableau is Cantorian tends to 0 as n increases. Therefore Theorem 10 should come as a surprise. Paradoxically, the infinite tableau T described in Theorem 10 is Cantorian. This of course is due to the rather stringent condition imposed on L !

THEOREM 11. *If T is as in the statement of Theorem 10 with $s = 2$ then* Perm(T) $= [0,1] \setminus L$.

Proof. By Theorem 10 the tableau T is Cantorian, thus we have that Perm(T) $\subseteq [0,1] \setminus L$. The equality is established as follows: let x be a number in $[0,1] \setminus L$. We show how to construct a permutation $\pi \in S_{\mathbf{N}}$ such that

$$x = 0.a^1_{\pi(1)} a^2_{\pi(2)} \cdots .$$

Writing x^j for the j-th digit of x we define

$$\pi(j) = \min\{i \mid a^j_i = x^j \text{ and } \forall j' < j : i \neq \pi(j')\} .$$

First we show that $\pi(j)$ is well defined: since L contains $\mathbf{Q}$ there is an infinity of rows i with $a^j_i = x^j$. On the other hand, there can be only a finite number of rows i with $i = \pi(j')$ for some $j' < j$. Thus, the set of which we take the minimum is indeed nonempty.

It is clear from the definition that π is one-to-one. Hence it remains to show that for all i there is a j such that $\pi(j) = i$. Note that there is an infinity

of columns j with $a_i^j = x^j$, because otherwise $x + l_i$ would be rational, where $l_i = a_i^1 a_i^2 \cdots$. This in turn cannot be the case, since l_i is in L and x is not.

Suppose now that for all j we have $\pi(j) \neq i$. It follows that $\pi(j) < i$ for all columns j with $a_i^j = x^j$. Since π is one-to-one this can be true for only a finite number of columns, thus contradicting the assumption. □

By choosing the set of algebraic numbers in the unit interval for L, we obtain the following corollary:

COROLLARY 12. *If the rows of T consist of all the algebraic numbers in the unit interval represented in base 2, then* Perm(T) *is exactly the set of all transcendental numbers in the unit interval.*

The condition $s = 2$ was necessary in Theorem 11 and Corollary 12. Indeed, for $s \geq 3$ there exist transcendental numbers with missing digits such as the Liouville numbers

$$\sum_{n \geq 0} s^{-n!}.$$

Therefore, Theorem 10 shows that for $s \geq 3$, Perm(T) cannot contain all transcendental numbers.

It is however true for all $s \geq 2$ that Perm(T) contains uncountably many transcendental numbers. Indeed, suppose l_i is the list of all the algebraic numbers as above and t_i is a list of some countable set of transcendental numbers. We show how to construct a permutation π such that

$$a^1_{\pi(1)} a^2_{\pi(2)} \cdots$$

is different from every t_i: note that any l_i and t_j differ from each other in infinitely many positions. Let i_1 be the first position where l_1 differs from t_1. We choose $\pi(1) = i_1$ and we set $\pi(j) = j - 1$ for all $1 < j < i_1$. Clearly $a^{i_1}_{\pi(1)} \neq t_1^{i_1}$. Now we proceed iteratively: when i_k is known, we choose i_{k+1} as the first position after i_k where l_{i_k+1} differs from t_{k+1}. Then we define $\pi(i_k + 1) = i_{k+1}$ and $\pi(j) = j - 1$ for all j such that $i_k + 1 < j < i_{k+1}$. π permutes each set $\{i_k + 1, i_k + 2, \ldots, i_{k+1}\}$, so it is indeed in $S_{\mathbf{N}}$ and $a^{i_{k+1}}_{\pi(i_k+1)} \neq t^{i_{k+1}}_{k+1}$.

We can in fact show the stronger result that the set of real numbers not contained in Perm(T) has measure 0. This is a consequence of the following theorem.

THEOREM 13. *Let T be an infinite tableau containing the s-expansions of a countable dense subset L of the unit interval. Then the measure of* Perm(T) *is* 1.

Proof. We will show that the probability that $x \notin \mathrm{Perm}(T)$ is less than ϵ for any $\epsilon > 0$. Recall that the probability that an n node random directed graph $D_{n,1/s}$ (where each possible edge has probability $1/s$ of being present) is Hamiltonian tends to 1 as n tends to infinity [6]. Define n_i for $i > 0$ to be the first n such that this probability is greater than $1 - \epsilon/2^i$ and let $N_0 = 0$ and $N_i = \sum_{j=1}^{i} n_j$.

From the initial order of the rows l_i we construct a new order l'_i as follows: l'_{N_i+1} is the first l_j not already present in $l'_1, l'_2, \ldots, l'_{N_i}$. For $N_i + 2 \le j \le N_{i+1}$, elements in positions $N_i + 1, \ldots, N_{i+1}$ of l'_j are chosen randomly and independently and l'_j is chosen as the first l_k with these elements and not already present in $l'_1, , l'_2, \ldots, l'_{j-1}$. Such an l'_j exists since L is dense. Because l_j is chosen at the latest in the j-th step, this defines a reordering of the rows.

Now we consider the probability that x can be obtained as the diagonal of a permutation of the rows l'_i fixing each of the sets $\{N_i+1, N_i+2, \ldots, N_{i+1}\}$. Let ${}^iv$ be the vector consisting of digits $N_i + 1, N_i + 2, \ldots, N_{i+1}$ of x. Let ${}^iB$ be the square Boolean matrix whose entry ${}^iB^k_j$ is true if and only if ${}^iM^k_j = {}^iv^k$. We claim that ${}^iB$ has each entry true with probability $1/s$ and that all these probabilities are independent: this is true for the first row of ${}^iB$ because x was chosen randomly, and for all other rows because the corresponding elements of ${}^iM$ were random.

Now ${}^iB$ is the adjacency matrix of a graph $D_{n_i,1/s}$ and we know that this graph has probability less than $\epsilon/2^i$ of not being Hamiltonian. If all the graphs are Hamiltonian there is a permutation of the rows l'_i that consists of a cycle on each of the sets $\{N_i + 1, N_i + 2, \ldots, N_{i+1}\}$ and produces x on the diagonal. Hence

$$\Pr[x \notin \mathrm{Perm}(T)] < \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon. \qquad \square$$

A similar approach allows us to consider tableaux whose rows consist of infinite sequences on a finite set $\{0, 1, \ldots, s-1\}$. Rational numbers would be replaced by ultimately periodic sequences and algebraic numbers would then be replaced by s-automatic sequences [1, 4, 5]. The results from Theorem 10 on remain valid with the obvious modifications.

7. OUTLOOK

Following the remark after the definition of Cantorian tableaux in Section 2, define an equivalence relation on the set of $n \times n$ tableaux as follows: let T' be equivalent to T, if it is obtained from T by a combination of permuting rows or columns or replacing all entries of a column by their image under any bijection on the alphabet. It might be interesting to count the number of resulting equivalence classes.

Taking into account the situation for base 2 in Theorem 10, it might also be interesting to consider those tableaux T where $\mathrm{Perm}(T) \cap L$ equals a given set, or has a given cardinality.

Finally, we could have defined "bi-Cantorian" tableaux as those where $\mathrm{Perm}(T)$ is disjoint both from the set of row-words and column-words. We chose our initial definition guided by Cantor's work. Needless to say it might well be interesting to extend our discussion to bi-Cantorian tableaux. For example, an argument very similar to the one given at the beginning of Section 5 shows that there are at least $2^{\lfloor n/2 \rfloor^2}$ bi-Cantorian tableaux of size $n \times n$ over the alphabet $\{a, b\}$.

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