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interpreting the former as elementary symmetric functions in certain roots of G (or their squares).

Ehresmann [E] investigated the topology of complex Grassmann manifolds (and other hermitian symmetric spaces) by studying the algebra of K -invariant differential forms on them ($K = U(N)$ for $X = G(m, n)$). This relies on the fact that the invariant forms are harmonic for the natural hermitian structure on X , which implies that the ring of all such forms is isomorphic to $H^*(X)$. Kostant [K1] [K2] later found analogues of these results for arbitrary (generalized) flag manifolds. The representation theory used to determine the K -invariant forms in this program does not directly relate the multiplicities $c_{\lambda\mu}^{\nu}$ in equations (1) and (2). Note however that the cited works of É. Cartan and Ehresmann were used by Chern in his fundamental paper on the characteristic classes of complex manifolds [Ch1]. More recently, Stoll [St] used fiber integration to study the algebra of invariant forms on the Grassmannian, but his work does not address the question posed in the Introduction.

Following [SGA6], [V] and [Be], the isomorphism between $\mathrm{gr}R(G)$ and $\mathrm{Pol}(\mathfrak{g})^G$ in §6 may be used to construct the Chern-Weil (or characteristic) homomorphism in algebraic geometry. Let $P \rightarrow X$ be a principal G -bundle over a smooth algebraic variety X and let $CH^*(X)$ denote the Chow group of algebraic cycles on X modulo rational equivalence. The Grothendieck group $K(X)$ of vector bundles on X is a λ -ring, with the λ -operations induced by exterior powers. According to [SGA6, Exp. XIV], the graded ring $\mathrm{gr}K(X) \otimes \mathbf{R}$ is canonically isomorphic to the real Chow ring $CH_{\mathbf{R}}^*(X) = CH^*(X) \otimes \mathbf{R}$. There is a natural λ -ring homomorphism $\pi: \mathcal{R}(G) \rightarrow K(X)$, defined by sending a representation $G \rightarrow \mathrm{GL}(E)$ to the associated vector bundle $P \times_G E$ over X . The characteristic homomorphism is the induced map

$$\mathrm{gr}(\pi)_{\mathbf{R}}: \mathrm{Pol}(\mathfrak{g})^G \longrightarrow CH_{\mathbf{R}}^*(X).$$

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