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interpreting the former as elementary symmetric functions in certain roots of G (or their squares).

Ehresmann [E] investigated the topology of complex Grassmann manifolds (and other hermitian symmetric spaces) by studying the algebra of K-invariant differential forms on them ($K = U(N)$ for $X = G(m,n)$). This relies on the fact that the invariant forms are harmonic for the natural hermitian structure on X, which implies that the ring of all such forms is isomorphic to $H^*(X)$. Kostant [K1] [K2] later found analogues of these results for arbitrary (generalized) flag manifolds. The representation theory used to determine the K-invariant forms in this program does not directly relate the multiplicities $c^{\nu}_{\lambda\mu}$ in equations (1) and (2). Note however that the cited works of É. Cartan and Ehresmann were used by Chern in his fundamental paper on the characteristic classes of complex manifolds [Ch1]. More recently, Stoll [St] used fiber integration to study the algebra of invariant forms on the Grassmannian, but his work does not address the question posed in the Introduction.

Following [SGA6], [V] and [Be], the isomorphism between $\mathrm{gr}\,\mathrm{R}(G)$ and $\mathrm{Pol}(\mathfrak{g})^G$ in §6 may be used to construct the Chern-Weil (or characteristic) homomorphism in algebraic geometry. Let $P \to X$ be a principal G-bundle over a smooth algebraic variety X and let $CH^*(X)$ denote the Chow group of algebraic cycles on X modulo rational equivalence. The Grothendieck group $K(X)$ of vector bundles on X is a λ-ring, with the λ-operations induced by exterior powers. According to [SGA6, Exp. XIV], the graded ring $\mathrm{gr}\,K(X)\otimes\mathbf{R}$ is canonically isomorphic to the real Chow ring $CH^*_{\mathbf{R}}(X) = CH^*(X)\otimes\mathbf{R}$. There is a natural λ-ring homomorphism $\pi\colon \mathcal{R}(G) \to K(X)$, defined by sending a representation $G \to \mathrm{GL}(E)$ to the associated vector bundle $P \times_G E$ over X. The characteristic homomorphism is the induced map

$$\mathrm{gr}(\pi)_{\mathbf{R}}\colon\ \mathrm{Pol}(\mathfrak{g})^G \longrightarrow CH^*_{\mathbf{R}}(X)\,.$$

REFERENCES

[Be] BEAUVILLE, A. Chern classes for representations of reductive groups. Preprint (2001): math.AG/0104031.

[B1] BOREL, A. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2) 57* (1953), 115–207.

[B2] —— Topology of Lie groups and characteristic classes. *Bull. Amer. Math. Soc. 61* (1955), 397–432.

[BH] BOREL, A. and F. HIRZEBRUCH. Characteristic classes and homogeneous spaces, I, II, III. *Amer. J. Math. 80* (1958), 459–538; *81* (1959), 315–382; *82* (1960), 491–504.

[BT] BOTT, R. and L. W. TU. *Differential Forms in Algebraic Topology.* Graduate Texts in Mathematics *82*. Springer-Verlag, Berlin–New York, 1982.

[CLL] CARRÉ, C., A. LASCOUX and B. LECLERC. Turbo-straightening for decomposition into standard bases. *Internat. J. Algebra Comput. 2* (1992), 275–290.

[C] CARRELL, J. Chern classes of the Grassmannians and Schubert calculus. *Topology 17* (1978), 177–182.

[Ca] CARTAN, É. Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces. *Annales de la Société Polonaise de Mathématique 8* (1929), 181–225. Reprinted in *Œuvres Complètes*, Part I, vol. 2, 1081–1125. Gauthier-Villars, Paris, 1952.

[Car] CARTAN, H. *a.* Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie. In: *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, 15–27.

b. La transgression dans un groupe de Lie et dans un espace fibré principal. *ibid.*, 57–71. Georges Thone, Liège; Masson, Paris, 1951. Reprinted in [GuS].

[Ch1] CHERN, S. S. Characteristic classes of Hermitian manifolds. *Ann. of Math. (2) 47* (1946), 85–121.

[Ch2] —— Differential geometry of fiber bundles. In: *Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950*, vol. 2, 397–411. Amer. Math. Soc., Providence, 1952.

[Ch3] —— *Topics in Differential Geometry.* The Institute for Advanced Study, Princeton, N. J., 1951. Reprinted in *Selected Papers*, vol. IV. Springer-Verlag, 1989, 331–397.

[Ch4] —— *Complex Manifolds* (mimeographed notes). University of Chicago, Chicago, 1956.

[Ch5] —— *Complex Manifolds Without Potential Theory.* Van Nostrand Math. Studies *15*. Van Nostrand, Princeton, 1968.

[Cl] CLAUSEN, M. Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups, I. *Adv. Math. 33* (1979), 161–191.

[DKR] DÉSARMÉNIEN, J., J. P. S. KUNG and G.-C. ROTA. Invariant theory, Young bitableaux, and combinatorics, *Adv. Math. 27* (1978), 63–92.

[D] DUPONT, J. L. *Curvature and Characteristic Classes.* Lecture Notes in Mathematics *640*. Springer-Verlag, Berlin–New York, 1978.

[E] EHRESMANN, C. Sur la topologie de certains espaces homogènes. *Ann. of Math. 35* (1934), 396–443.

[Fa] FARKAS, D. R. Multiplicative invariants. *L'Enseignement Math. (2) 30* (1984), 141–157.

[F1] FULTON, W. *Young Tableaux*. London Math. Soc. Student Texts *35*. Cambridge University Press, Cambridge, 1997.

[F2] —— Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bull. Amer. Math. Soc. (N.S.) 37* (2000), 209–249.

[FH] FULTON, W. and J. HARRIS. *Representation Theory. A First Course*. Graduate Texts in Math. *129*. Springer-Verlag, New York, 1991.

[FL] FULTON, W. and S. LANG. *Riemann-Roch Algebra*. Grundlehren der mathem. Wiss. *277*. Springer-Verlag, New York, 1985.

[G1] GIAMBELLI, G. Z. Risoluzione del problema degli spazi secanti. *Mem. R. Accad. Sci. Torino (2) 52* (1902), 171–211.

[G2] —— Alcune proprietà delle funzioni simmetriche caratteristiche. *Atti Torino 38* (1903), 823–844.

[GK] GRABMEIER, J. and A. KERBER. The evaluation of irreducible polynomial representations of the general linear groups and of the unitary groups over fields of characteristic 0. *Acta Appl. Math. 8* (1987), 271–291.

[Gr] GREEN, J. A. Schur algebras and general linear groups. In: *Groups – St. Andrews 1989*, Vol. 1, 155–210. London Math. Soc. Lecture Note Ser. *159*. Cambridge Univ. Press, Cambridge, 1991.

[GHV] GREUB, W., S. HALPERIN and R. VANSTONE. *Connections, Curvature, and Cohomology. I: de Rham Cohomology of Manifolds and Vector Bundles; II: Lie Groups, Principal Bundles, and Characteristic Classes; III: Cohomology of Principal Bundles and Homogeneous Spaces*. Pure and Applied Mathematics *47*. Academic Press, New York–London, 1972.

[GS] GRIFFITHS, P. and W. SCHMID. Locally homogeneous complex manifolds. *Acta Math. 123* (1969), 253-302.

[GuS] GUILLEMIN, V. and S. STERNBERG. *Supersymmetry and Equivariant de Rham Theory, with an appendix containing two reprints by Henri Cartan*. Springer-Verlag, Berlin, 1999.

[HP] HODGE, W. V. D. and D. PEDOE. *Methods of Algebraic Geometry*. Cambridge Univ. Press, Cambridge, 1952.

[Ho] HORROCKS, G. On the relation of S-functions to Schubert varieties. *Proc. London Math. Soc. (3) 7* (1957), 265–280.

[Hu] HUSEMOLLER, D. *Fibre Bundles*, second edition. Graduate Texts in Mathematics *20*. Springer-Verlag, 1975.

[J] JACOBI, C. G. J. De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum. *Crelle's Journal 22* (1841), 360–371. Reprinted in *Gesammelte Werke 3*, 439–452. Chelsea, New York, 1969.

[KT] KOIKE, K. and I. TERADA. Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n. *J. Algebra 107* (1987), 466–511.

[K1] KOSTANT, B. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2) 74* (1961), 329–387.

[K2] —— Lie algebra cohomology and generalized Schubert cells. *Ann. of Math. (2) 77* (1963), 72–144.

[KM] KOSTANT, B. and P. W. MICHOR. The generalized Cayley map from an algebraic group to its Lie algebra. In: *The Orbit Method in Geometry and Physics (Marseille, 2000)*, 259–296. Progr. Math. *213*. Birkhäuser Boston, Boston, 2003.

[Len] LENART, C. The combinatorics of Steenrod operations on the cohomology of Grassmannians. *Adv. in Math. 136* (1998), 251–283.

[Les] LESIEUR, L. Les problèmes d'intersection sur une variété de Grassmann. *C.R. Acad. Sci. Paris 225* (1947), 916–917.

[M] MACDONALD, I. *Symmetric Functions and Hall Polynomials*, second edition. Clarendon Press, Oxford, 1995.

[Ma] MARTIN, S. *Schur Algebras and Representation Theory.* Cambridge Tracts in Math. *112*. Cambridge University Press, Cambridge, 1993.

[MS] MILNOR, J. W. and J. D. STASHEFF. *Characteristic Classes.* Annals of Mathematics Studies *76*. Princeton University Press, Princeton, 1974.

[NR] NARASIMHAN, M. S. and S. RAMANAN. Existence of universal connections I, II. *Amer. J. Math. 83* (1961), 563–572; *85* (1963), 223–231.

[O] OSSE, A. λ-structures and representation rings of compact connected Lie groups. *J. Pure Appl. Algebra 121* (1997), 69–93.

[P] PIERI, M. Sul problema degli spazi secanti. Nota 1ª. *Rend. Ist. Lombardo (2) 26* (1893), 534–546.

[PR] PRAGACZ, P. and J. RATAJSKI. Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach. *Compositio Math. 107* (1997), 11–87.

[Sa] SAMELSON, H. Topology of Lie groups. *Bull. Amer. Math. Soc. 58* (1952), 2–37.

[S1] SCHUR, I. Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen. Dissertation, Berlin, 1901. Reprinted in *Gesammelte Abhandlungen 1*, 1-72. Springer-Verlag, Berlin–New York, 1973.

[S2] —— Über die rationalen Darstellungen der allgemeinen linearen Gruppe. *Sitzungsberichte der Preuss. Akad. der Wiss.* (1927), 58–75. Reprinted in *Gesammelte Abhandlungen 3*, 68-85. Springer-Verlag, Berlin–New York, 1973.

[SGA6] BERTHELOT, P., A. GROTHENDIECK and L. ILLUSIE. *Théorie des intersections et théorème de Riemann-Roch.* Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6). Lecture Notes in Math. *225*. Springer-Verlag, Berlin–New York, 1971.

[St] STOLL, W. *Invariant Forms on Grassmann Manifolds.* Annals of Math. Studies *89*. Princeton University Press, Princeton, 1977.

[T] TAMVAKIS, H. Arithmetic intersection theory on flag varieties. *Math. Ann. 314* (1999), 641–665.

[V] VISTOLI, A. Characteristic classes of principal bundles in algebraic intersection theory. *Duke Math. J. 58* (1989), 299–315.

[W] WEIL, A. Géométrie différentielle des espaces fibrés. Manuscript (1949). *Œuvres Scientifiques/Collected Papers*, Vol. I, 422–436. Springer-Verlag, 1979.

[We] WEYL, H. *The Classical Groups: Their Invariants and Representations*. Princeton University Press, Princeton (N. J.), 1939.

[Z] ZELOBENKO, D. P. *Compact Lie Groups and Their Representations*. Translations of Mathematical Monographs *40*. Amer. Math. Soc., Providence, 1973.

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