Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 50 (2004)
Heft: 1-2: L'enseignement mathématique

Artikel: A genus formula for some plane curves
Autor: Kang, Ming-Chang
DOI: https://doi.org/10.5169/seals-2644

L'Enseignement Mathématique, t. 50 (2004), p. 139–146

A GENUS FORMULA FOR SOME PLANE CURVES

by Ming-Chang KANG*)

ABSTRACT. Let k be an algebraically closed field with $\operatorname{char} k = 0$, and Γ_0 be the irreducible affine curve defined by the equation $f(x) = g(y)$. We determine the genus of Γ_0 in terms of the numerical data of f and g.

1. INTRODUCTION

Throughout this article k is an algebraically closed field with $\operatorname{char} k = 0$, unless otherwise specified. (See the remarks at the end of this note for weaker assumptions about the field k.)

Let $f(T)$, $g(T) \in k[T]$ be non-constant monic polynomials, let Γ_0 be the affine plane curve defined by the equation $f(x) = g(y)$, and let Γ be the projective plane curve associated to Γ_0, i.e. Γ is defined by $X_0^N f(X_1/X_0) = X_0^N g(X_2/X_0)$, where $N = \max\{\deg f, \deg g\}$, $x = X_1/X_0$, $y = X_2/X_0$. Assume that $f(x) - g(y) \in k[x,y]$ is an irreducible polynomial. The purpose of this note is to find the genus of (the normalization of) the plane algebraic curve Γ in terms of the numerical data of $f(x)$ and $g(y)$. The class of algebraic curves Γ_0 of the form $f(x) = g(y)$ includes hyper-elliptic curves, Fermat curves, some curves arising in arithmetic questions and coding theory, etc. (see Theorem 1 and [Pr]). It is desirable to find an explicit formula for the genus of such a plane curve.

Let $m = \deg f$, $n = \deg g$, $d = \gcd\{m,n\}$. Define $R = \{a \in k : f'(a) = 0\}$, $S = \{b \in k : g'(b) = 0\}$, $\operatorname{Sing}(\Gamma_0) = \{(a,b) \in R \times S : f(a) = g(b)\}$. We list all the elements of $\operatorname{Sing}(\Gamma_0)$ as $(a_1,b_1), (a_2,b_2), \ldots, (a_l,b_l)$. (It may happen that $l = 0$, i.e. that $\operatorname{Sing}(\Gamma_0) = \varnothing$.) For each $(a_i,b_i) \in \operatorname{Sing}(\Gamma_0)$, write

*) Partially supported by the National Science Council, Republic of China.

$f(x) - g(y) = (x - a_i)^{m_i} f_1(x) - (y - b_i)^{n_i} g_1(y)$, where $f_1(a_i) g_1(b_i) \neq 0$. Define $d_i = \gcd\{m_i, n_i\}$.

The genus of the curve Γ_0 was estimated by Davenport, Lewis and Schinzel in connection with integral solutions of $f(x) = g(y)$, viz.

THEOREM 1 ([DLS], Theorem 1). *Assume that $f(T), g(T) \in \mathbf{Z}[T]$ and that there are distinct values $a_1, a_2, \ldots, a_t \in R$ such that $t \geq [m/2]$ with $f(x) - g(y) = (x - a_i)^{m_i} f_1(x) - h_i(y)$, where $f_1(a_i) \neq 0$ and $h_i(y) = 0$ has no multiple roots for $1 \leq i \leq t$. Then $f(x) - g(y) \in \mathbf{C}[x, y]$ is irreducible. Moreover, if $n \neq 2$ and $(m, n) \neq (3, 3)$, then the genus of Γ_0 is positive and the equation $f(x) = g(y)$ has finitely many integral solutions.*

The main result of this article is the following

THEOREM 2. *Let $f(T), g(T) \in k[T]$ and let $\Gamma_0, \Gamma, m, n, d, m_i, n_i, d_i$ be defined as above. Assume that Γ_0 is an irreducible curve. Then the genus of Γ is equal to*

$$\{(m-1)(n-1) + 1 - d\}/2 - \sum_{1 \leq i \leq l} \{(m_i - 1)(n_i - 1) - 1 + d_i\}/2 .$$

Note that, if Γ is a non-singular curve, it is necessary that $|m - n| \leq 1$, i.e. all the points of Γ lying on the infinite line $X_0 = 0$ are non-singular points. If Γ is non-singular and $m = n$, the above formula reduces to the well-known formula $(m-1)(m-2)/2$; if Γ is non-singular and $|m - n| = 1$, the genus of Γ becomes $(m-1)(n-1)/2$.

In [Mi], p. 74 (Problem III.2 D) it was suggested to find a genus formula for the cyclic covering $y^n = f(x)$ of the affine line by applying Hurwitz's formula, although no explicit formula was exhibited there. Instead we will prove Theorem 2 by using Plücker's formula (see Theorem 3 in Section 2).

We emphasize that in Theorem 2 one must assume that the curve $f(x) - g(y) = 0$ is irreducible. This assumption is not a very serious restriction. For, as a consequence of the classification of finite simple groups, if $f(T), g(T)$ are indecomposable polynomials, then $f(x) - g(y)$ is irreducible except when (i) $g(T) = f(aT + b)$ for some $a, b \in k$, or (ii) $\deg f = \deg g = 7, 11, 13, 15, 21$, or 31. (See [Fe], Theorem 1.1. A polynomial $f(T) \in k[T]$ is *indecomposable* if, wherever $f(T) = f_1(f_2(T))$ for polynomials $f_1(T), f_2(T) \in k[T]$, we have $\deg f_i = 1$ for $i = 1$ or 2.)

2. THE PROOF

We recall Plücker's formula first. (See [Se], p. 65, Formula (19); [Hi]; [Ca], pp. 111–113. See also the remark in [BK], Theorem 5, p. 614.)

THEOREM 3. *Let Γ be a projective plane curve of degree N. Then the genus of Γ is equal to*

$$(N-1)(N-2)/2 - \sum \delta_P ,$$

where P runs over the singular points of Γ, $\mathcal{O}_P$ denotes the local ring at P, $\overline{\mathcal{O}}_P$ is the normalization of $\mathcal{O}_P$, and $\delta_P = \dim(\overline{\mathcal{O}}_P/\mathcal{O}_P)$.

LEMMA 4. *Let $k[[x,y]]$ be the power series ring, m and n be positive integers, $A = k[[x,y]]/(x^m - y^n)$, and $\bar{A}$ the normalization of A. If $d = \gcd\{m,n\}$, then $\dim_k(\bar{A}/A) = \{(m-1)(n-1) - 1 + d\}/2$.*

Proof. STEP 1. If $d = \gcd\{m,n\} = 1$, then $\dim(\bar{A}/A) = (m-1)(n-1)/2$.

Note that $A = k[[x,y]]/(x^m - y^n) \simeq k[[t^n, t^m]] \hookrightarrow k[[t]] \simeq \bar{A}$. Since A is a complete intersection, A is a Gorenstein local ring and we can apply [Se], p. 72, Proposition 7. It follows that $\dim(\bar{A}/A) = n_P/2$, where $\langle t^{n_P} \rangle$ is the conductor ideal of $\bar{A} \cong k[[t]]$ into $A \cong k[[t^n, t^m]]$.

On the other hand, the conductor ideal $\langle t^M \rangle$ of $k[[t]]$ into $k[[t^n, t^m]]$ is simply that given by $M = \min\{p \in \mathbf{N} :$ for any $q \geq p$, q can be written as $nx + my$ for some non-negative integers $x, y\}$. It is not difficult to determine this non-negative integer M. In fact, $M = (m-1)(n-1)$. (See [NZ], p. 107, Exercise 9.) Hence the result.

STEP 2. Now consider the case $d = \gcd\{m,n\} \geq 2$. Write $m = dr, n = ds$.

Let ζ be a primitive d-th root of unity. Note that $x^m - y^n = (x^r)^d - (y^s)^d = \prod_{1 \leq i \leq d}(x^r - \zeta^i y^s)$. All factors $x^r - \zeta^i y^s$ are relatively prime irreducible elements in $k[[x,y]]$ because $k[[x,y]]/(x^r - \zeta^i y^s) \cong k[[t^s, t^r]]$ is a subring of $k[[t]]$. (Note that the factor ζ^i in $x^r - \zeta^i y^s$ can be absorbed into y^s.)

Let I_i be the prime ideal in $k[[x,y]]$ generated by $x^r - \zeta^i y^s$. Since $I_1 I_2 \cdots I_d = I_1 \cap I_2 \cap \cdots \cap I_d$, it follows that

$$A = k[[x,y]]/\prod_{1 \leq i \leq d}(x^r - \zeta^i y^s) = k[[x,y]]/I_1 \cdots I_d \hookrightarrow \prod_{1 \leq j \leq d} B_j ,$$

where $B_j = k[[x,y]]/I_j$.

On the other hand, the set of non-zero divisors of $A = k[[x,y]]/I_1 \cap \cdots \cap I_d$ is the image of $S = k[[x,y]] \setminus I_1 \cup \cdots \cup I_d$. Thus the total quotient ring of A is $S^{-1}A$, which is just the total quotient ring of $\prod_{1 \le j \le d} B_j$. It follows that the normalization of A is $\prod_{1 \le j \le d} \bar{B}_j$, where $\bar{B}_j$ is the normalization of B_j.

To sum up, $\dim(\bar{A}/A) = \dim(\prod_{1 \le j \le d} B_j/A) + \sum_{1 \le j \le d} \dim(\bar{B}_j/B_j)$. Note that $\dim(\bar{B}_j/B_j) = (r-1)(s-1)/2$ by Step 1 since $\gcd\{r,s\} = 1$. It remains to prove that $\dim(\prod_{1 \le j \le d} B_j/A) = d(d-1)rs/2$.

STEP 3. We will prove that $\dim(\prod_{1 \le j \le d} B_j/A) = d(d-1)rs/2$.

Define $C_j = k[[x,y]]/I_j \cap I_{j+1} \cap \cdots \cap I_d$ and $D_j = k[[x,y]]/(I_{j-1}, I_j \cap \cdots \cap I_d)$ for $2 \le j \le d$. We get the following short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B_1 \times C_2 & \longrightarrow & D_2 & \longrightarrow & 0, \\
0 & \longrightarrow & C_2 & \longrightarrow & B_2 \times C_3 & \longrightarrow & D_3 & \longrightarrow & 0, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \longrightarrow & C_{d-1} & \longrightarrow & B_{d-1} \times C_d & \longrightarrow & D_d & \longrightarrow & 0.
\end{array}$$

It follows that $A \subset B_1 \times C_2 \subset B_1 \times B_2 \times C_3 \subset \cdots \subset B_1 \times B_2 \times \cdots \times B_d$ since $C_d = B_d$. Hence

$$\dim\Big(\prod_{1 \le j \le d} B_j/A\Big) = \dim\big((B_1 \times C_2)/A\big) + \sum_{3 \le j \le d} \dim((B_{j-1} \times C_j)/C_{j-1}) = \sum_{2 \le j \le d} \dim(D_j).$$

Note that

$$D_j = k[[x,y]]/(I_{j-1}, I_j \cap \cdots \cap I_d) = k[[x,y]]/\Big(x^r - \zeta^{j-1}y^s, \prod_{j \le i \le d}(x^r - \zeta^i y^s)\Big)$$
$$= k[[x,y]]/(x^r - \zeta^{j-1}y^s, y^{s(d-j+1)}).$$

Hence $\dim(D_j) = rs(d-j+1)$. Thus $\sum_{2 \le j \le d} \dim(D_j) = d(d-1)rs/2$. □

PROOF OF THEOREM 2

The singular points P of Γ are either points belonging to $\mathrm{Sing}(\Gamma_0)$ or points lying on the infinite line $X_0 = 0$. We shall compute δ_P and apply Theorem 3.

STEP 1. $P = (a_i, b_i) \in \operatorname{Sing}(\Gamma_0)$.

Let $\delta_i = \dim(\bar{R}_i/R_i)$ where R_i is the local ring at (a_i, b_i) and $\bar{R}_i$ the normalization of R_i, for $1 \leq i \leq l$.

Since δ_i is invariant under completion by [Se], p. 59, Formula (3), it suffices to compute $\delta_i = \dim(\bar{A}_i/A_i)$, where $A_i \simeq k[[x,y]]/(x^{m_i} - y^{n_i})$ is the completion of R_i and $\bar{A}_i$ is the normalization of A_i. (For, considering the case $i = 1$, we may assume that $a_1 = b_1 = 0$ and write $f(x) = x^{m_1} f_1(x)$, $g(y) = y^{n_1} g_1(y)$ where $f_1(0) g_1(0) \neq 0$. The elements $f_1(x)$, $g_1(y)$ are units in $k[[x,y]]$ and can be written as β^{m_1}, γ^{n_1} for some $\beta, \gamma \in k[[x,y]]$. Define $X = x\beta$ and $Y = y\gamma$. Then $A_1 \simeq k[[x,y]]/(g(y) - f(x)) \simeq k[[X,Y]]/(Y^{n_1} - X^{m_1})$.) By Lemma 4, it follows that $\dim(\bar{A}_i/A_i) = \{(m_i - 1)(n_i - 1) - 1 + d_i\}/2$.

STEP 2. If $|m - n| \leq 1$, then the projective curve is non-singular except for those points belonging to $\operatorname{Sing}(\Gamma_0)$. It is easy to check that $(N-1)(N-2)/2 = \{(m-1)(n-1) + 1 - d\}/2$, where $N = \max\{m, n\}$.

STEP 3. Consider the case $m \geq n + 2$. (The case $m \leq n - 2$ is similar and will be omitted.)

Consider the homogenized polynomial equation $X_0^m\, g(X_2/X_0) = X_0^m f(X_1/X_0)$ where $x = X_1/X_0$, $y = X_2/X_0$ and we shall write $f(x) = \prod_{1 \leq i \leq m}(x + \lambda_i)$, $g(y) = \prod_{1 \leq j \leq n}(y + \rho_j)$. The only singular point of Γ other than those belonging to $\operatorname{Sing}(\Gamma_0)$ is $P = (0 : 0 : 1)$. Let $z = X_0/X_2$, $w = X_1/X_2$. The dehomogenized polynomial becomes $z^{m-n} \prod_{1 \leq j \leq n}(1 + \rho_j z) = \prod_{1 \leq i \leq m}(w + \lambda_i z)$. It suffices to compute $\delta_P = \dim(\bar{A}/A)$, where

$$A = k[w,z]_{(w,z)} / \Big(z^{m-n} \prod_{1 \leq j \leq n} (1 + \rho_j z) - \prod_{1 \leq i \leq m} (w + \lambda_i z)\Big),$$

the local ring of Γ at the point P. Note that the multiplicity at the point P is $m - n$.

STEP 4. The element $\prod_{1 \leq j \leq n}(1 + \rho_j z)$ is a unit in A. Call it ϵ.

In the local ring A, consider the relation: $\epsilon z^{m-n} = \prod_{1 \leq i \leq m}(w + \lambda_i z)$. Define $u = z/w$ in the quotient field of A. The above relation becomes $\epsilon u^{m-n} = w^n \prod_{1 \leq i \leq m}(1 + \lambda_i u)$.

Write $\prod_{1 \leq i \leq m}(1 + \lambda_i u) = \sum_{0 \leq i \leq m} a_i u^i$, where $a_i \in k$ and $a_0 = 1$. Then we get $\epsilon u^{m-n} - w^n \sum_{m-n \leq i \leq m} a_i u^i = w^n \sum_{0 \leq i \leq m-n-1} a_i u^i$. Hence

$$u^{m-n}\Big(\epsilon - \sum_{m-n \leq i \leq m} a_i w^n u^{i-m+n}\Big) = \sum_{0 \leq i \leq m-n-1} a_i w^n u^i .$$

As

$$\epsilon - \sum_{m-n\le i\le m} a_i w^n u^{i-m+n} = \epsilon - a_{m-n}w^n - a_{m-n+1}w^{n-1}z - \cdots - a_m z^m$$

is a unit in A, we find

(1) $$u^{m-n} + \alpha_1 u^{m-n-1} + \alpha_2 u^{m-n-2} + \cdots + \alpha_{m-n} = 0\,,$$

where $\alpha_i \in w^n A$. It follows that $u \in \bar{A}$.

Clearly, $A \subset B \subset \bar{A}$, where

$$B = A[u] = k[w,u]_{(w,u)}/\big(\epsilon u^{m-n} - w^n \prod_{1\le i\le m}(1+\lambda_i u)\big)\,.$$

Thus $\delta_P = \dim(\bar{A}/A) = \dim(B/A) + \dim(\bar{A}/B)$.

STEP 5. We claim that $\dim(B/A) = (m-n)(m-n-1)/2$.

First we will show that

(2) $$B = A + \sum_{0\le i<j\le m-n-1} k\cdot w^i u^j\,.$$

Let C be the completion of A. Then $C = k[[w,z]]/\big(\epsilon z^{m-n} - \prod_{1\le i\le m}(w+\lambda_i z)\big)$. Since $B/A = A[u]/A$ is a finite-dimensional vector space over k, it is naturallly isomorphic to $(A[u]/A)\otimes_A C$. Thus (2) is equivalent to

(3) $$C[u] = C + \sum_{0\le i<j\le m-n-1} k\cdot w^i u^j\,.$$

To check the validity of (3) it suffices to consider whether $z^i u^j$ (where $0\le j\le m-n-1$) belongs to the right-hand-side of (3), because of Formula (1) and the relation $uw = z$. We will prove it by induction on j. If $j=0$, it is trivial. Now assume $j\ge 1$. In case $j \le m-n-i-1$, then $z^i u^j = w^i u^{i+j}$ with $i+j\le m-n-1$ as required. If $j\ge m-n-i$, then $z^i u^j = w^i u^{i+j}$ and $i+j\ge m-n$. Using (1) we find that u^{i+j} is a linear combination of $w^n u^l$ with coefficients in A, where $0\le l\le m-n-1$. It follows that, after modulo C, $z^i u^j$ is a linear combination of u^{l-n-i} (if $l-n-i>0$) where $l\le m-n-1$. Note that $l-n-i\le m-2n-i-1<j$. Thus we have reduced $z^i u^j$ to terms of lower exponents.

We shall show that the $w^i u^j$ are linearly independent in B/A where $0\le i<j\le m-n-1$. Suppose $0 = \sum_{i<j} a_{ij} w^i u^j$ in B/A where $a_{ij}\in k$ are not all zero. Hence $\sum_{i<j} a_{ij} w^i u^j = \varphi \in A$. Define $p = \max\{j-i : a_{ij}\ne 0\}$. Multiply the relation $\sum_{i<j} a_{ij} w^i u^j = \varphi$ by w^p. We get $\sum_{j-i=p} a_{ij} z^j = w\psi$ for some $\psi\in A$. Since $1\le j\le m-n-1$, it follows that the multiplicity of the point P is $\le m-n-1$, a contradiction.

STEP 6. Note that $\bar{A}$ is also the normalization of B. We will compute $\dim(\bar{A}/B)$.

By [Se], p. 59, Formula (3), the question can again be reduced to the complete case. Let D be the completion of B. Then

$$D = k[[w,u]]/\Big(\epsilon u^{m-n} - w^n \prod_{1\le i\le m} (1+\lambda_i u)\Big).$$

Find $\alpha \in D$ such that $\alpha^{m-n} = \epsilon^{-1}\prod_{1\le i\le m}(1+\lambda_i u)$. Define $U = u/\alpha$. Then $D \simeq k[[w,U]]/(U^{m-n} - w^n)$. Now we can apply Lemma 4 to get $\dim(\bar{D}/D) = \{(n-1)(m-n-1) - 1 + d\}/2$.

STEP 7. Finally we find that $\delta_P = \dim(\bar{A}/A) = \dim(B/A) + \dim(\bar{A}/B) = \{(m-1)(m-n-1) - 1 + d\}/2$. Thus

$$(N-1)(N-2)/2 - \delta_P = \{(m-1)(n-1) + 1 - d\}/2$$

because $N = \max\{m,n\} = m$. This completes the proof of Theorem 2.

REMARKS.

(1) From the above proof, it is clear that Theorem 2 remains valid if $\operatorname{char} k = p > 0$ and p doesn't divide $mn\prod_{1\le i\le l} m_i n_i$.

(2) Similarly, if p is a prime number and the affine curve is defined by $y^p = \prod_{1\le i\le l}(x-\lambda_i)^{m_i}$ such that the λ_i are distinct, $1 \le m_i < p$ and p doesn't divide $\sum_{1\le i\le l} m_i$, then Theorem 2 (and its proof for this case) remains valid no matter what $\operatorname{char} k$ may be. Note that the latter assumption can always be achieved. For, if we denote $\sum_{1\le i\le l} m_i$ by m and suppose that $m = pr$, we may assume that $\lambda_1 = 0$. Divide both sides of the equation by x^m. Consider the new variables $u = 1/x$, $v = y/x^r$.

(3) On the other hand, if we assume that k is a perfect field (such that (i) $p \nmid mn\prod_{1\le i\le l} m_i n_i$ if $\operatorname{char} k = p > 0$, or (ii) p is a prime number and the affine curve is defined by $y^p = \prod_{1\le i\le l}(x-\lambda_i)^{m_i}$ with ...) but not algebraically closed, then Theorem 2 is true because we can extend the constant field k to its algebraic closure at the beginning of the proof without affecting the genus by [Ch], p. 99.

REFERENCES

[BK] BRIESKORN, E. and H. KNÖRRER. *Plane Algebraic Curves*. Birkhäuser, Basel, 1986.

[Ca] CASAS-ALVERO, E. *Singularities of Plane Curves*. London Math. Soc. Lecture Note Series *276*. Cambridge Univ. Press, Cambridge, 2000.

[Ch] CHEVALLEY, C. *Introduction to the Theory of Algebraic Functions of One Variable.* Amer. Math. Soc., Providence, 1951.

[DLS] DAVENPORT, H., D. J. LEWIS and A. SCHINZEL. Equations of the form $f(x) = g(y)$. *Quart. J. Math. Oxford Ser. (2) 12* (1961), 304–312.

[Fe] FEIT, W. Some consequences of the classification of finite simple groups. In: *Proc. Symp. Pure Math. 37* (1980), 175–181.

[Hi] HIRONAKA, H. On the arithmetic genera and the effective genera of algebraic curves. *Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 30* (1957), 177–195.

[Mi] MIRANDA, R. *Algebraic Curves and Riemann Surfaces.* Graduate Studies in Math. *5.* Amer. Math. Soc., Providence, 1995.

[NZ] NIVEN, I. and H. S. ZUCKERMAN. *An Introduction to the Theory of Numbers,* 2nd ed. Wiley, New York, 1966.

[Pr] PRETZEL, O. *Codes and Algebraic Curves.* Clarendon Press, Oxford, 1998.

[Se] SERRE, J.-P. *Algebraic Groups and Class Fields.* Springer GTM *117.* Springer-Verlag, Berlin, 1988.

(Reçu le 13 novembre 2003)

Ming-Chang Kang

Department of Mathematics
National Taiwan University
Taipei, Taiwan
e-mail: kang@math.ntu.edu.tw