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Since the torus knot groups are the only knot groups with non-trivial center the groups of all other knots cannot be lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.

9.2. So we now consider knots which are not torus knots; their groups can be lattices in $\mathrm{PSL}_2(\mathbf{C})$ only. As for $\mathrm{PSL}_2(\mathbf{C})$, it is the isometry group of hyperbolic 3-space \mathbf{H}^3 . The knot complement C is a Haken 3-manifold with zero-Euler characteristic. It is atoroidal precisely if the respective knot does not have a companion; we use here “companion” in the sense of non-trivial companion. Indeed the boundary torus of a regular neighborhood of a companion would be a non-boundary-parallel incompressible surface in C . Thus by Thurston’s Hyperbolization Theorem the interior of the complement of a knot without companion can be given a hyperbolic structure with finite volume coming from \mathbf{H}^3/G . In other words that knot group G is a lattice in $\mathrm{PSL}_2(\mathbf{C})$ and the interior of C can be identified with the open manifold \mathbf{H}^3/G .

Concerning the notion of companion see [Ro, p. 111]. For concepts related to the Hyperbolization Theorem and to properties of 3-manifolds we refer to [K].

9.3. If the knot has a companion then the knot complement is not atoroidal and its interior does not admit a hyperbolic structure [K, Cor.4.63]. It follows that G cannot be a lattice in (the only remaining possibility) $\mathrm{PSL}_2(\mathbf{C})$. We sketch the proof: If G is such a lattice then \mathbf{H}^3/G is a $K(G, 1)$ -manifold as well, thus homotopy equivalent to C . The homotopy equivalence can be turned into a diffeomorphism mapping \mathbf{H}^3/G to the interior of the knot complement C which thus would receive a hyperbolic structure with finite volume.

THEOREM 9.1. *Torus knot groups are lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$. As for other groups of knots, those of knots without companion are lattices in $\mathrm{PSL}_2(\mathbf{C})$, and those of knots with companion are not lattices in any connected Lie group.*

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