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## LINEAR FUNCTIONAL EQUATIONS AND SHAPIRO'S CONJECTURE

by M. LACZKOVICH \*)

ABSTRACT. We investigate the functional equation

$$\sum_{i=1}^n a_i(y) f_i(x + b_i(y)) = h(y) \quad (x, y \in \mathbf{R}),$$

where  $a_i, f_i$ , and  $h$  are complex valued functions defined on  $\mathbf{R}$ , and  $b_1, \dots, b_n$  are real valued functions such that  $b_i - b_j$  is not constant on any interval. We prove that under mild regularity conditions (e.g., if  $a_1, \dots, a_n$  are nonvanishing functions of bounded variation,  $b_1, \dots, b_n$  are  $d$ -convex and  $f_1, \dots, f_n$  are measurable) the functions  $f_1, \dots, f_n$  must be exponential polynomials. We also show that the continuity of the functions  $b_i$  and  $f_i$  implies the same conclusion, subject to Shapiro's conjecture on exponential polynomials with constant coefficients.

### 1. INTRODUCTION

The functional equation

$$(1) \quad \sum_{i=1}^n a_i(y) f_i(x + b_i(y)) = h(y)$$

has been studied extensively, and several papers have been devoted to the regularity properties of the solutions  $f_1, \dots, f_n$ . In [12] and [1] it is shown that if the functions  $a_i$  and  $b_i$  are smooth enough and if  $f_1, \dots, f_n$  are locally integrable then  $f_1, \dots, f_n$  are necessarily  $C^\infty$  functions. In this paper we show that under mild regularity conditions on the functions  $a_i$  and  $b_i$ , the functions  $f_i$  must be exponential polynomials, even if we only assume measurability instead of local integrability.

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We shall say that the function  $\phi: [a, b] \rightarrow \mathbf{R}$  is *d-convex* if it can be written as the difference of two continuous convex functions. It is easy to see that  $\phi: [a, b] \rightarrow \mathbf{R}$  is d-convex and Lipschitz if and only if  $\phi$  is absolutely continuous and if the function  $\phi'$  (defined on the set of points where  $\phi$  is differentiable) is of bounded variation. Clearly, every  $C^2$  function is d-convex.

A function  $f: \mathbf{R} \rightarrow \mathbf{C}$  is said to be an exponential polynomial if  $f(x) = \sum_{i=1}^n p_i(x) e^{\alpha_i x}$ , where  $p_1, \dots, p_n$  are polynomials with complex coefficients and  $\alpha_1, \dots, \alpha_n$  are complex numbers.

**THEOREM 1.** *Let  $J$  be a nondegenerate interval, and suppose that the functions  $a_i: J \rightarrow \mathbf{C}$  and  $b_i: J \rightarrow \mathbf{R}$  ( $i = 1, \dots, n$ ) have the following properties.*

- (i) *Each of the functions  $a_1, \dots, a_n$  is nonvanishing on  $J$  and is of bounded variation;*
- (ii) *each of the functions  $b_1, \dots, b_n$  is d-convex on  $J$ ; and*
- (iii) *the function  $b_i - b_j$  is not constant on any subinterval of  $J$  for every  $1 \leq i < j \leq n$ .*

*Let  $h: J \rightarrow \mathbf{C}$  be an arbitrary function, and let  $f_1, \dots, f_n$  be complex valued measurable functions on  $\mathbf{R}$  such that (1) holds for almost every  $(x, y) \in \mathbf{R} \times J$ . Then each of the functions  $f_1, \dots, f_n$  equals an exponential polynomial almost everywhere.*

The necessity of condition (iii) is shown by the fact that any function  $f: \mathbf{R} \rightarrow \mathbf{C}$  satisfies

$$f(x) + f(x + y) - f(x + \max(y, 0)) - f(x + \min(y, 0)) = 0$$

for every  $(x, y) \in \mathbf{R}^2$ .

We can formulate many similar statements by imposing different conditions on the functions involved. Two of the most interesting variants are the following.

**STATEMENT M.** *Suppose that the functions  $a_i: J \rightarrow \mathbf{C}$  and  $b_i: J \rightarrow \mathbf{R}$  ( $i = 1, \dots, n$ ) are measurable,  $a_i$  is nonvanishing on  $J$  for every  $i = 1, \dots, n$ , and  $b_i - b_j$  is not constant on any set of positive measure for every  $1 \leq i < j \leq n$ . Let  $h: J \rightarrow \mathbf{C}$  be an arbitrary function, and let  $f_1, \dots, f_n$  be complex valued measurable functions on  $\mathbf{R}$  such that (1) holds for almost every  $(x, y) \in \mathbf{R} \times J$ . Then each of the functions  $f_1, \dots, f_n$  equals an exponential polynomial almost everywhere.*

STATEMENT C. Suppose that the functions  $a_i: J \rightarrow \mathbf{C}$  and  $b_i: J \rightarrow \mathbf{R}$  ( $i = 1, \dots, n$ ) are continuous,  $a_i$  is nonvanishing on  $J$  for every  $i = 1, \dots, n$ , and  $b_i - b_j$  is not constant on any subinterval of  $J$  for every  $1 \leq i < j \leq n$ . Let  $h: J \rightarrow \mathbf{C}$  be an arbitrary function, and let  $f_1, \dots, f_n$  be complex valued continuous functions on  $\mathbf{R}$  such that (1) holds for every  $(x, y) \in \mathbf{R} \times J$ . Then each of the functions  $f_1, \dots, f_n$  is an exponential polynomial.

We do not know if Statements M and C are true or not. We shall prove, however, that Statement C is a consequence of Shapiro's conjecture.

Let  $\mathcal{R}$  denote the set of difference operators of the form

$$\Delta f = \sum_{i=1}^n a_i \cdot f(x + b_i),$$

where  $a_i$  and  $b_i$  are complex. If we define addition in the obvious way and multiplication by  $(\Delta_1 \Delta_2)f = \Delta_1(\Delta_2 f)$  then we obtain a commutative ring with identity. (In fact, what we obtain is the complex group ring over the additive group of  $\mathbf{C}$ .) The one-to-one correspondence between  $\Delta$  and its characteristic function

$$(2) \quad \sum_{i=1}^n a_i e^{b_i z}$$

is an isomorphism between  $\mathcal{R}$  and the ring  $\mathcal{E}$  of all exponential polynomials with constant coefficients. The units of the ring  $\mathcal{E}$  are the functions of the form  $a \cdot e^{bz}$ , where  $a \neq 0$ . The exponential polynomial (2) is called simple if the frequencies  $b_1, \dots, b_n$  are pairwise commensurable; that is, if  $b_i/b_j$  is rational whenever  $b_j \neq 0$ . By a theorem of J. F. Ritt [9], every nonzero and non-unit exponential polynomial has a factorization of the form  $f_1 \cdot \dots \cdot f_s \cdot g_1 \cdot \dots \cdot g_t$ , where  $f_1, \dots, f_s$  are simple, the frequencies of  $f_i$  and  $f_j$  are noncommensurable if  $i \neq j$ , and each  $g_k$  is irreducible. The factorization is unique up to unit multiples.

H. S. Shapiro conjectured in [11] that if two exponential polynomials have infinitely many common roots then they have a non-unit common divisor. As Shapiro remarked, the Lech-Mahler theorem implies the conjecture in the special case when one of the exponential polynomials is simple. (See [11, p. 18] and [8].) The conjecture in its general form is still open.

Recall that a topological space  $Y$  is Baire if every meager subset of  $Y$  has empty interior.

**THEOREM 2.** *Suppose that Shapiro's conjecture is true. Let  $Y$  be a topological space such that  $Y^n$  is Baire, and let the functions  $a_i: Y \rightarrow \mathbf{C}$  and  $b_i: Y \rightarrow \mathbf{R}$  ( $i = 1, \dots, n$ ) satisfy the following conditions:  $a_i$  is nonvanishing on  $Y$ ,  $b_i$  is continuous for every  $i = 1, \dots, n$ , and  $b_i - b_j$  is not constant on any nonempty open subset of  $Y$  for every  $1 \leq i < j \leq n$ . Let  $h: Y \rightarrow \mathbf{C}$  be an arbitrary function, and let  $f_1, \dots, f_n$  be complex valued continuous functions on  $\mathbf{R}$  such that (1) holds for every  $(x, y) \in \mathbf{R} \times Y$ . Then each of the functions  $f_1, \dots, f_n$  is an exponential polynomial.*

## 2. TRANSLATION INVARIANT CLOSED SUBSPACES OF $C(\mathbf{R})$

Let  $C(\mathbf{R})$  denote the space of complex valued continuous functions on  $\mathbf{R}$  endowed with the topology of uniform convergence on compact intervals. In the proof of Theorems 1 and 2 we shall use L. Schwartz's celebrated theorem stating that spectral synthesis holds in  $C(\mathbf{R})$ ; that is, if  $L$  is any translation invariant closed subspace of  $C(\mathbf{R})$  then the set of exponential polynomials contained in  $L$  form a dense subset of  $L$ . (See [10], [5] and [6].) Schwartz's theorem immediately implies that if  $L$  is a finite dimensional invariant subspace of  $C(\mathbf{R})$  then  $L$  consists of exponential polynomials. We prove Theorem 1 – at least in the case when  $h \equiv 0$  – by showing that the functions  $f_i$  must belong to finite dimensional invariant subspaces of  $C(\mathbf{R})$ .

**LEMMA 3.** *Let  $L$  be a translation invariant closed subspace of  $C(\mathbf{R})$ . Suppose that*

- (i) *there exists a nonzero difference operator  $\Delta$  such that  $\Delta f = 0$  for every  $f \in L$ , and*
- (ii) *every element of  $L$  is locally Lipschitz.*

*Then  $L$  is finite dimensional.*

*Proof.* Let  $\Delta f(x) = \sum_{j=1}^p a_j f(x + b_j)$  ( $f \in C(\mathbf{R})$ ), where  $a_1, \dots, a_p$  are nonzero and  $b_1 < \dots < b_p$ . If  $L$  is not finite dimensional then, by Schwartz's theorem, the spectrum  $\text{sp}(L) = \{\lambda \in \mathbf{C} : e^{\lambda x} \in L\}$  is infinite. If  $\lambda \in \text{sp}(L)$  then  $\Delta e^{\lambda z} = 0$  by (i), and thus  $E(\lambda) = 0$ , where  $E(z) = \sum_{j=1}^p a_j e^{b_j z}$ . That is,  $\text{sp}(L)$  is a subset of the set of roots of  $E(z)$ , and hence the elements of  $\text{sp}(L)$  can be listed as  $\lambda_n = \sigma_n + it_n$  ( $n = 1, 2, \dots$ ), where  $|\lambda_n| \rightarrow \infty$ . Now

$$\lim_{\text{Re } z \rightarrow \infty} \frac{E(z)}{e^{b_p z}} = a_p \quad \text{and} \quad \lim_{\text{Re } z \rightarrow -\infty} \frac{E(z)}{e^{b_1 z}} = a_1,$$

and hence there is a positive number  $K$  such that  $E(\sigma + it) \neq 0$  if  $|\sigma| > K$ . Therefore  $|\sigma_n| \leq K$  for every  $n$ . Since  $|\lambda_n| \rightarrow \infty$ , it follows that  $|t_n| \rightarrow \infty$ .

We select a sequence  $n_1, n_2, \dots$  as follows. Let  $n_1$  be chosen such that  $|t_{n_1}| > 20\pi K$ . If  $n_1, \dots, n_{k-1}$  have been selected then we choose  $n_k$  with the following properties:  $|t_{n_k}| > 20^k \pi K$ , and

$$(3) \quad \left| \exp\left(\frac{\pi \lambda_{n_j}}{t_{n_k}}\right) - 1 \right| < \frac{1}{10^k}$$

for every  $j < k$ . This defines the indices  $n_k$  for every  $k$ . Now we put  $f(x) = \sum_{j=1}^{\infty} 10^{-j} e^{\lambda_{n_j} x}$  for every  $x \in \mathbf{R}$ . Since  $|e^{\lambda_n x}| \leq e^{K \cdot |x|}$  for every  $n$  and for every  $x \in \mathbf{R}$ , it follows that the series is uniformly convergent on compact intervals, and thus  $f$  is an element of  $L$ . We shall prove that  $f$  is not locally Lipschitz at 0. By (ii), this will provide a contradiction, proving that  $\text{sp}(L)$  must be finite.

We have  $f(\pi/t_{n_k}) - f(0) = \sum_{j=1}^{\infty} 10^{-j} A_k^j$ , where

$$A_k^j = \exp\left(\frac{\pi \sigma_{n_j} + i\pi t_{n_j}}{t_{n_k}}\right) - 1.$$

Now  $|A_k^j| < 10^{-k}$  for every  $j < k$  by (3),

$$|A_k^k| = \left| \exp\left(\frac{\pi \sigma_{n_k} + i\pi}{t_{n_k}}\right) - 1 \right| = \exp\left(\frac{\pi \sigma_{n_k}}{t_{n_k}}\right) + 1 > 1,$$

and

$$|A_k^j| \leq \exp\left(\frac{\pi \sigma_{n_j}}{t_{n_k}}\right) + 1 \leq \exp\left(\frac{\pi K}{t_{n_k}}\right) + 1 < 3$$

for every  $j > k$ . Therefore,

$$\begin{aligned} |f(\pi/t_{n_k}) - f(0)| &\geq \frac{1}{10^k} |A_k^k| - \sum_{j=1}^{k-1} \frac{1}{10^j} |A_k^j| - \sum_{j=k+1}^{\infty} \frac{1}{10^j} |A_k^j| \\ &\geq \frac{1}{10^k} - \sum_{j=1}^{k-1} \frac{1}{10^j} \cdot \frac{1}{10^k} - \sum_{j=k+1}^{\infty} \frac{1}{10^j} \cdot 3 \\ &\geq \frac{1}{2 \cdot 10^k}. \end{aligned}$$

Thus

$$\left| \frac{f(\pi/t_{n_k}) - f(0)}{(\pi/t_{n_k})} \right| \geq \frac{1}{2 \cdot 10^k} \cdot \frac{20^k \pi K}{\pi} = 2^{k-1} K$$

for every  $k$ , proving that  $f$  is not locally Lipschitz.  $\square$

REMARK. Condition (i) cannot be omitted from Lemma 3: there are infinite dimensional translation invariant closed subspaces of  $C(\mathbf{R})$  that only contain locally Lipschitz functions. One can show, for example, that if  $\lambda_n$  is a sequence of real numbers converging to infinity fast enough, then every element of the closed subspace  $L$  generated by the exponentials  $e^{\lambda_n x}$  is real analytic, but  $L$  is not finite dimensional.

### 3. REDUCTION

Let  $G$  be an Abelian group, and let  $\mathcal{R}_G$  denote the algebra of difference operators of the form  $\Delta f = \sum_{i=1}^n a_i \cdot f(x + b_i)$  ( $a_i \in \mathbf{C}$ ,  $b_i \in G$ ). The translation operator  $T_b$  ( $b \in G$ ) is defined by  $T_b f = f(x + b)$ . Clearly, every difference operator is the linear combination of translation operators. We shall use determinants of the form

$$(4) \quad \begin{vmatrix} \Delta_{1,1} & \cdots & \Delta_{1,n-1} & f_1 \\ \vdots & & \vdots & \vdots \\ \Delta_{n,1} & \cdots & \Delta_{n,n-1} & f_n \end{vmatrix},$$

where  $\Delta_{i,j} \in \mathcal{R}_G$  ( $i = 1, \dots, n$ ;  $j = 1, \dots, n-1$ ), and  $f_j: G \rightarrow \mathbf{C}$  ( $i = 1, \dots, n$ ). These determinants are defined as follows. In the formal expansion of (4) every term is of the form  $\pm p_1 \cdots p_n$ , where exactly one of the factors  $p_i$  is a function and the other factors are difference operators. Rearranging the factors such that the function comes last we obtain an expression of the form  $\Delta f$ , defining a map from  $G$  into  $\mathbf{C}$ . Then we define (4) as the sum of these functions.

Let  $Y$  be a nonempty set, and suppose that the functions  $f_j: G \rightarrow \mathbf{C}$ ,  $a_j: Y \rightarrow \mathbf{C}$ ,  $b_j: Y \rightarrow G$  ( $j = 1, \dots, n$ ) and  $h: Y \rightarrow \mathbf{C}$  satisfy

$$(5) \quad \sum_{j=1}^n a_j(y) \cdot f_j(x + b_j(y)) = h(y)$$

for every  $(x, y) \in G \times Y$ . We can write (5) as

$$(6) \quad \sum_{j=1}^n a_j(y) T_{b_j(y)} f_j = h(y).$$

Let  $y_1, \dots, y_n \in Y$  be arbitrary elements. Substituting  $y_1, \dots, y_n \in Y$  into (6) we obtain  $\sum_{j=1}^n a_j(y_i) T_{b_j(y_i)} f_j = h(y_i)$  ( $i = 1, \dots, n$ ).

Then we have

$$(7) \quad \begin{vmatrix} a_1(y_1)T_{b_1(y_1)} & \cdots & a_{n-1}(y_1)T_{b_{n-1}(y_1)} & \sum_{j=1}^n a_j(y_1)T_{b_j(y_1)}f_j \\ \vdots & & \vdots & \vdots \\ a_1(y_n)T_{b_1(y_n)} & \cdots & a_{n-1}(y_n)T_{b_{n-1}(y_n)} & \sum_{j=1}^n a_j(y_n)T_{b_j(y_n)}f_j \end{vmatrix} \\ = \begin{vmatrix} a_1(y_1)T_{b_1(y_1)} & \cdots & a_{n-1}(y_1)T_{b_{n-1}(y_1)} & a_n(y_1)T_{b_n(y_1)}f_n \\ \vdots & & \vdots & \vdots \\ a_1(y_n)T_{b_1(y_n)} & \cdots & a_{n-1}(y_n)T_{b_{n-1}(y_n)} & a_n(y_n)T_{b_n(y_n)}f_n \end{vmatrix};$$

this can be justified in the same way as for determinants with numerical entries. The left hand side of (7), as a function of  $x$ , is constant, since each entry of its last column is constant. If we denote the value of the left hand side by  $H(y) = H(y_1, \dots, y_n)$  and expand the right hand side of (7), then we obtain the following

LEMMA 4. *Suppose that the functions  $f_j: G \rightarrow \mathbf{C}$ ,  $a_j: Y \rightarrow \mathbf{C}$ ,  $b_j: Y \rightarrow G$  ( $j = 1, \dots, n$ ) and  $h: Y \rightarrow \mathbf{C}$  satisfy (5) for every  $(x, y) \in G \times Y$ . Put  $N = n!$ . Then there are functions  $A_i: Y^n \rightarrow \mathbf{C}$  and  $B_i: Y^n \rightarrow G$  ( $i = 1, \dots, N$ ) and  $H: Y^n \rightarrow \mathbf{C}$  such that*

(i) *we have*

$$(8) \quad \sum_{i=1}^N A_i(y) f_n(x + B_i(y)) = H(y)$$

*for every  $x \in G$  and  $y \in Y^n$ ;*

(ii) *for every  $i = 1, \dots, N$  there are indices  $j_1, \dots, j_n$  such that  $A_i(y) = \pm a_{j_1}(y_1) \cdots a_{j_n}(y_n)$  for every  $y = (y_1, \dots, y_n) \in Y^n$ ;*

(iii) *for every  $i = 1, \dots, N$  there are indices  $k_1, \dots, k_n$  such that  $B_i(y) = b_{k_1}(y_1) + \dots + b_{k_n}(y_n)$  for every  $y = (y_1, \dots, y_n) \in Y^n$ ;*

(iv) *if  $b_{j_1} - b_{j_2}$  is not constant for every  $1 \leq j_1 < j_2 \leq n$ , then  $B_{i_1} - B_{i_2}$  is not constant for every  $1 \leq i_1 < i_2 \leq N$ ;*

(v) *if  $h \equiv 0$  then  $H \equiv 0$ .*

REMARK. We shall need the following 'almost everywhere' version of Lemma 4 in the special case when  $G = \mathbf{R}$  and  $Y$  is a subinterval of  $\mathbf{R}$ . Suppose that the measurable functions  $f_j: \mathbf{R} \rightarrow \mathbf{C}$ ,  $a_j: Y \rightarrow \mathbf{C}$ ,  $b_j: Y \rightarrow \mathbf{R}$  ( $j = 1, \dots, n$ ) and  $h: Y \rightarrow \mathbf{C}$  satisfy (5) for a.e.  $(x, y) \in \mathbf{R} \times Y$  with respect to the Lebesgue measure  $\lambda_2$ . Then there are functions  $A_i: Y^n \rightarrow \mathbf{C}$  and  $B_i: Y^n \rightarrow \mathbf{R}$  ( $i = 1, \dots, N$ ) and  $H: Y^n \rightarrow \mathbf{C}$  satisfying (ii)–(v) of Lemma 4

and such that (8) holds for a.e.  $(x, y) \in \mathbf{R} \times Y^n$  with respect to  $\lambda_{n+1}$ . The proof of this statement is the same as that of Lemma 4.

#### 4. REGULARITY OF SOLUTIONS

In this section we show that – under the conditions formulated in Theorem 1 – the measurable solutions of (1) are locally Lipschitz. We remark that by imposing more restrictive regularity conditions on the functions  $a_i$  and  $b_i$  (namely,  $a_i, b_i \in C^2$ ) this result could be deduced from a general theorem of A. Járαι [4]. Our result is based on the observation that if  $f$  is bounded measurable and  $g$  is of bounded variation then their convolution is Lipschitz. (See Lemma 7 below.)

LEMMA 5. *If  $g$  is a nonconstant  $d$ -convex function on  $J$  then there are a subinterval  $J_1 \subset J$  and a positive number  $\varepsilon$  such that  $g$  is strictly monotonic on  $J_1$ ; moreover, either  $g'(x) \geq \varepsilon$  for a.e.  $x \in J_1$  or  $g'(x) \leq -\varepsilon$  for a.e.  $x \in J_1$ .*

*Proof.* Since  $g$  is absolutely continuous and nonconstant, the set  $H = \{x \in J : g'(x) \neq 0\}$  is of positive measure. Also,  $g'$  is of bounded variation in every closed subinterval of the interior of  $J$ , and thus  $g'$  is continuous almost everywhere. Consequently, there is a point  $x_0 \in H$  at which  $g'$  is continuous. Let  $0 < \varepsilon < |g'(x_0)|/2$  be fixed, and choose a small neighbourhood  $J_1$  of  $x_0$  such that  $|g'(x) - g'(x_0)| < \varepsilon$  whenever  $x \in J_1$  and  $g'$  exists. It is clear that  $J_1$  and  $\varepsilon$  satisfy the requirements.  $\square$

LEMMA 6. *Let  $g: J \rightarrow \mathbf{R}$  be differentiable a.e. on the bounded interval  $J$ , and suppose that  $g'(x) \neq 0$  for a.e.  $x \in J$ . Then (i)  $g^{-1}(H)$  is null for every null set  $H \subset \mathbf{R}$ , and (ii) for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\lambda(H) < \delta$  implies  $\lambda(g^{-1}(H)) < \varepsilon$ .*

*Proof.* Let  $\lambda(H) = 0$ , and suppose that  $A = g^{-1}(H)$  is of positive outer measure. Since  $g'(x) \neq 0$  for a.e.  $x \in A$ , we can select a positive number  $\varepsilon$  and a set  $B \subset A$  of positive outer measure such that either  $g'(x) > \varepsilon$  or  $g'(x) < -\varepsilon$  for every  $x \in B$ . We may assume that  $g' > \varepsilon$  on  $B$ , since otherwise we replace  $g$  by  $-g$ . Then there is a positive integer  $n$  and there is a subset  $C \subset B$  of positive outer measure such that  $(g(y) - g(x))/(y - x) > \varepsilon$  for every  $x \in C$  and for every  $y \in J$  with  $0 < |y - x| < 1/n$ . Let  $L$  be a

subinterval of  $J$  such that  $|L| < 1/n$  and  $\lambda(C \cap L) > 0$ . Put  $D = C \cap L$ ; then  $\lambda(D) > 0$  and  $|g(y) - g(x)| \geq \varepsilon|y - x|$  for every  $x, y \in D$ . In particular,  $g$  is one-to-one on  $D$ . Let  $g(D) = E$  and  $f = (g|D)^{-1}$ . Then  $E \subset H$  and  $f$  maps  $E$  onto  $D$ . Also,  $f$  is Lipschitz on  $E$ , since  $|f(u) - f(v)| \leq |u - v|/\varepsilon$  holds for every  $u, v \in E$ . Since  $\lambda(E) \leq \lambda(H) = 0$ , this implies  $\lambda(D) = 0$ , a contradiction. This proves (i).

Suppose that (ii) is false. Then there is an  $\varepsilon > 0$  and there are sets  $H_n$  such that  $\lambda(H_n) < 1/n^2$  and  $\lambda(g^{-1}(H_n)) \geq \varepsilon$  for every  $n = 1, 2, \dots$ . We may assume that the sets  $H_n$  are open. Since  $g$  is measurable (in fact,  $g$  is continuous a.e.), it follows that the sets  $g^{-1}(H_n)$  are measurable. Let  $H = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} H_n$ . Then  $\lambda(H) = 0$ , and

$$\lambda(g^{-1}(H)) = \lambda\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} g^{-1}(H_n)\right) \geq \liminf_{n \rightarrow \infty} \lambda(g^{-1}(H_n)) \geq \varepsilon,$$

which contradicts (i). □

LEMMA 7. *Let  $U$  be of bounded variation on the interval  $[a, b]$ . Let  $I$  be a compact interval, and let  $f$  be measurable and bounded on the interval  $I + [a, b]$ . Then the function*

$$F(x) = \int_a^b f(x+y)U(y) dy \quad (x \in I)$$

*is Lipschitz on  $I$ .*

*Proof.* Let  $I + [a, b] = [c, d]$ , and put  $\Phi(x) = \int_c^x f(t) dt$  ( $x \in [c, d]$ ). Then  $\Phi$  is a Lipschitz function such that  $\Phi' = f$  a.e. on  $I + [a, b]$ . Denoting  $\Phi(y+x)$  by  $T_x\Phi(y)$  we obtain

$$\begin{aligned} (9) \quad F(x) &= \int_a^b U \cdot (T_x\Phi)' dy = \int_a^b U d(T_x\Phi) = [U \cdot T_x\Phi]_a^b - \int_a^b T_x\Phi dU \\ &= U(b) \cdot \Phi(x+b) - U(a) \cdot \Phi(x+a) - \int_a^b T_x\Phi dU. \end{aligned}$$

If  $|\Phi(x_1) - \Phi(x_2)| \leq K \cdot |x_1 - x_2|$  for every  $x_1, x_2$  then we have

$$\begin{aligned} \left| \int_a^b T_{x_1}\Phi dU - \int_a^b T_{x_2}\Phi dU \right| &= \left| \int_a^b (T_{x_1}\Phi - T_{x_2}\Phi) dU \right| \\ &\leq K \cdot |x_1 - x_2| \cdot V(U; [a, b]), \end{aligned}$$

and thus the function  $x \mapsto \int_a^b T_x\Phi dU$  is Lipschitz. Then, by (9), so is  $F$ . □

LEMMA 8. *Suppose that*

$$F(x) = \int_a^b c(y)f(x+g(y)) dy \quad (x \in I),$$

where

- $c: [a, b] \rightarrow \mathbf{C}$  is of bounded variation,
- $g: [a, b] \rightarrow \mathbf{R}$  is  $d$ -convex and Lipschitz,
- there is a positive number  $\varepsilon$  such that  $|g'(x)| \geq \varepsilon$  at every point  $x \in [a, b]$  where  $g'(x)$  exists,
- $I$  is a compact interval, and
- $f$  is measurable and bounded on the interval  $I + g([a, b])$ .

Then the function  $F$  is Lipschitz on  $I$ .

*Proof.* Since  $g'$  is of bounded variation, the oscillation of  $g'$  is less than  $2\varepsilon$  everywhere, except at the points of a finite set. Then, by  $|g'| \geq \varepsilon$  it follows that there is a subdivision  $a = a_0 < a_1 < \dots < a_n = b$  of  $[a, b]$  such that, for every  $i = 1, \dots, n$ ,  $g$  is strictly monotonic on  $[a_{i-1}, a_i]$ , and either  $g'(x) \geq \varepsilon$  for a.e.  $x \in [a_{i-1}, a_i]$  or  $g'(x) \leq -\varepsilon$  for a.e.  $x \in [a_{i-1}, a_i]$ . Let  $F_i(x) = \int_{a_{i-1}}^{a_i} c(y)f(x+g(y)) dy$  ( $x \in I$ ;  $i = 1, \dots, n$ ). Since  $F = F_1 + \dots + F_n$ , it is enough to show that each  $F_i$  is Lipschitz on  $I$ .

Let  $i$  be fixed. We may assume that  $g$  is strictly increasing on  $[a_{i-1}, a_i]$ ; the case when  $g$  is decreasing can be treated similarly. Let  $A = g(a_{i-1})$ ,  $B = g(a_i)$ , and let  $G$  denote the inverse of  $g|_{[a_{i-1}, a_i]}$ . Then  $G$  is absolutely continuous (in fact, Lipschitz) and strictly increasing on  $[A, B]$ . Since  $G' = 1/(g' \circ G)$ ,  $g' \geq \varepsilon$  and  $g'$  is of bounded variation on  $[A, B]$ , it follows that  $G'$  is also of bounded variation on  $[A, B]$ . Let  $U$  be an extension of  $(c \circ G) \cdot G'$  to  $[A, B]$  having finite variation. Then we have  $F_i(x) = \int_A^B f(x+u)U(u) du$  for every  $x \in I$ , and thus, by Lemma 7,  $F_i$  is Lipschitz on  $I$ .  $\square$

For every closed interval  $J$  and positive integer  $n$  we shall denote by  $\Phi_J^n$  the family of all functions of the form

$$A(y) = a_1(y_1) \cdots a_n(y_n) \quad (y = (y_1, \dots, y_n) \in J^n),$$

where  $a_1, \dots, a_n$  are complex valued nonvanishing functions of bounded variation defined on  $J$ . The set of the functions  $b_1(y_1) + \dots + b_n(y_n)$ , where  $b_i: J \rightarrow \mathbf{R}$  is a  $d$ -convex function on  $J$  for every  $i = 1, \dots, n$  will be denoted by  $\Psi_J^n$ .

By a subinterval of  $J^n$  we shall mean a set of the form  $J_1 \times \dots \times J_n$ , where  $J_1, \dots, J_n$  are nondegenerate subintervals of  $J$ .

LEMMA 9. Let  $A_i \in \Phi_J^n$  and  $B_i \in \Psi_J^m$  for every  $i = 1, \dots, N$ , and suppose that  $B_i - B_j$  is not constant on any subinterval of  $J^n$  for every  $1 \leq i < j \leq N$ . Let  $f_1, \dots, f_N$  be complex valued measurable functions on  $\mathbf{R}$  such that

$$(10) \quad \sum_{i=1}^N A_i(y) f_i(x + B_i(y)) = 0$$

for almost every  $(x, y) \in \mathbf{R} \times J^n$ . Then each of the functions  $f_1, \dots, f_N$  equals a locally Lipschitz function almost everywhere.

*Proof.* By symmetry, it is enough to show that  $f_1$  equals a locally Lipschitz function almost everywhere.

Let  $U$  denote the set of points  $(x, y) \in \mathbf{R} \times J^n$  for which (10) holds. Then  $(x - B_1(y), y) \in U$  for a.e.  $(x, y) \in \mathbf{R} \times J^n$ , and thus

$$\sum_{i=1}^N A_i(y) f_i(x + B_i(y) - B_1(y)) = 0$$

holds for a.e.  $(x, y) \in \mathbf{R} \times J^n$ . Therefore we may replace  $B_i$  by  $B_i - B_1$  for every  $i$ . After these replacements we find that  $B_1 \equiv 0$ .

Let  $A_i(y) = \prod_{k=1}^n a_{i,k}(y_k)$  and  $B_i(y) = \sum_{k=1}^n b_{i,k}(y_k)$ , where  $a_{i,k}: J \rightarrow \mathbf{C}$  is a nonvanishing function of bounded variation, and  $b_{i,k}: J \rightarrow \mathbf{R}$  is a  $d$ -convex function for every  $i = 1, \dots, N$  and  $k = 1, \dots, n$ . Since the functions  $a_{i,k}$  are continuous everywhere on  $J$  apart from a countable set, they have a common point of continuity  $x_0$ . As  $a_{i,k}(x_0) \neq 0$  for every  $i$  and  $k$ , there is an  $\eta > 0$  and there is a neighbourhood  $J_0$  of  $x_0$  such that  $|a_{i,k}(x)| > \eta$  for every  $i = 1, \dots, N$ ,  $k = 1, \dots, n$  and  $x \in J_0$ . Replacing  $J$  by  $J_0$  we may clearly assume that  $|a_{i,k}(x)| > \eta$  holds everywhere on  $J$  for every  $i$  and  $k$ . Then  $a_{i,k}/a_{1,k}$  is of bounded variation for every  $i$  and  $k$ , and thus  $A_i/A_1 \in \Phi_J^n$  for every  $i = 1, \dots, N$ . We replace  $A_i$  by  $A_i/A_1$  for every  $i$ ; then we have  $A_1 \equiv 1$  and

$$(11) \quad f_1(x) = - \sum_{i=2}^N A_i(y) f_i(x + B_i(y))$$

for a.e.  $(x, y) \in \mathbf{R} \times J^n$ .

Let  $1 < i \leq N$  and the subinterval  $J' \subset J$  be fixed. We claim that there is a  $k \in \{1, \dots, n\}$  and there is a subinterval  $J'' \subset J'$  such that  $b_{i,k}$  is not constant in every subinterval of  $J''$ . Indeed, otherwise we could find, successively, the intervals  $J' \supset J_1 \supset J_2 \supset \dots \supset J_n$  such that  $b_{i,k}$  is constant in  $J_k$  for every  $k = 1, \dots, n$ . Then  $B_i = B_i - B_1$  would be constant in  $(J_n)^n$ ,

contrary to the assumption. Applying this observation for every  $1 < i \leq N$  successively, we find a subinterval  $\bar{J} \subset J$  with the following property: for every  $1 < i \leq N$  there is a  $k(i) \in \{1, \dots, n\}$  such that  $b_{i,k(i)}$  is not constant in every subinterval of  $\bar{J}$ . Clearly, we may assume that  $J = \bar{J}$ . By taking another subinterval of  $J$ , we can suppose that each  $b_{i,k}$  is Lipschitz in  $J$ .

Applying Lemma 5,  $N-1$  times in succession, we find a positive  $\varepsilon$  and a subinterval  $J_1 \subset J$  such that, for every  $1 < i \leq N$ ,  $b_{i,k(i)}$  is strictly monotonic on  $J_1$ , and  $|b'_{i,k(i)}| \geq \varepsilon$  almost everywhere on  $J_1$ . Again, we may assume that  $J_1 = J$ . Then, by Lemma 6, we can find a positive number  $\delta$  such that  $\lambda\left(b_{i,k(i)}^{-1}(H)\right) < |J|/N$  whenever  $\lambda(H) < \delta$  and  $i = 2, \dots, N$ .

Let  $i \in \{2, \dots, N\}$  be arbitrary. We show that  $\lambda_n(B_i^{-1}(H)) < |J|^n/N$  for every  $H \subset \mathbf{R}$ ,  $\lambda(H) < \delta$ . We may suppose that  $H$  is open, and then so is  $B_i^{-1}(H)$ . If  $y_j \in J$  is fixed for every  $j \in \{1, \dots, n\} \setminus \{k(i)\}$  then

$$(y_1, \dots, y_n) \in B_i^{-1}(H) \iff b_{i,k(i)}(y_{k(i)}) \in H - \sum_{j \neq k(i)} b_{i,j}(y_j),$$

and thus

$$\lambda(\{y_{k(i)} : (y_1, \dots, y_n) \in B_i^{-1}(H)\}) = \lambda\left(b_{i,k(i)}^{-1}\left(H - \sum_{j \neq k(i)} b_{i,j}(y_j)\right)\right) < |J|/N,$$

since  $\lambda\left(H - \sum_{j \neq k(i)} b_{i,j}(y_j)\right) = \lambda(H) < \delta$ . Therefore, by Fubini's theorem, we obtain

$$\lambda_n(B_i^{-1}(H)) < |J|^{n-1} \cdot |J|/N = |J|^n/N,$$

as we stated.

We prove that  $f_1$  is locally essentially bounded. Let  $I$  be an arbitrary compact interval. Fubini's theorem implies that there is a set  $X \subset \mathbf{R}$  of full measure such that for every  $x \in X$ , (11) holds for a.e.  $y \in J^n$ . If  $K$  is large enough then the measure of each of the sets  $H_K^i = \{x \in I + B_i(J^n) : |f_i(x)| > K\}$  ( $i = 2, \dots, N$ ) is less than  $\delta$ . Therefore, by the choice of  $\delta$ , the set

$$E_x = \bigcup_{i=2}^N B_i^{-1}(H_K^i - x)$$

is of measure less than  $|J|^n$  for every  $x$ . Then the set  $J^n \setminus E_x$  is of positive measure for every  $x \in \mathbf{R}$ , and hence we can choose a point  $y_x \in J^n \setminus E_x$  for every  $x \in X$  such that (11) holds with  $y = y_x$ . Since  $x + B_i(y_x) \notin H_K^i$  for every  $i = 2, \dots, N$ , we have

$$|f_1(x)| \leq \sum_{i=2}^N \sup_{J^n} |A_i| \cdot K$$

for every  $x \in I \cap X$ . Since the interval  $I$  was arbitrary, it follows that  $f_1$  is locally essentially bounded. Clearly, the same is true for every  $f_i$ .

Now we show that  $f_1$  equals a locally Lipschitz function almost everywhere. By (11) we have

$$|J|^n \cdot f_1(x) = - \sum_{i=2}^N \int_{J^n} A_i(y) f_i(x + B_i(y)) d\lambda_n(y)$$

for a.e.  $x$ . Clearly, it is enough to show that

$$F_i(x) = \int_{J^n} A_i(y) f_i(x + B_i(y)) d\lambda_n(y) \quad (x \in \mathbf{R})$$

defines a locally Lipschitz function for every  $i = 2, \dots, N$ . Let  $i$  be fixed. Putting

$$u(z) = \prod_{j \neq k(i)} a_{i,j}(y_j) \quad (z = (y_1, \dots, y_{k(i)-1}, y_{k(i)+1}, \dots, y_n))$$

we have

$$(12) \quad F_i(x) = \int_{J^{n-1}} u(z) \cdot \left[ \int_J a_{i,k(i)}(t) \cdot f_i(x + d(z) + b_{i,k(i)}(t)) dt \right] d\lambda_{n-1}(z),$$

where  $d(z) = \sum_{j \neq k(i)} b_{i,j}(y_j)$ . By Lemma 8, the function

$$L(x) = \int_J a_{i,k(i)}(t) \cdot f_i(x + b_{i,k(i)}(t)) dt$$

is locally Lipschitz on  $\mathbf{R}$ . Since

$$F_i(x) = \int_{J^{n-1}} u(z) \cdot L(x + d(z)) d\lambda_{n-1}(z)$$

by (12), it follows that  $F_i$  is also locally Lipschitz. Indeed, let  $I$  be a compact interval. Since  $d$  is continuous on  $J^{n-1}$ , it follows that  $I' = I + d(J^{n-1})$  is also a compact interval. Let  $K$  be the Lipschitz constant of  $L$  on  $I'$ . If  $x_1, x_2 \in I$  and  $z \in J^{n-1}$  then  $x_1 + d(z), x_2 + d(z) \in I'$  and thus

$$\begin{aligned} |F_i(x_2) - F_i(x_1)| &\leq \int_{J^{n-1}} |u(z)| \cdot |L(x_2 + d(z)) - L(x_1 + d(z))| d\lambda_{n-1}(z) \\ &\leq K \cdot |x_2 - x_1| \cdot \int_{J^{n-1}} |u(z)| d\lambda_{n-1}, \end{aligned}$$

proving that  $F_i$  is locally Lipschitz.  $\square$

## 5. PROOF OF THEOREM 1

First we shall assume that the function  $h$  is identically zero. By symmetry, it is enough to show that  $f_n$  equals an exponential polynomial almost everywhere.

Suppose that the functions  $a_i, b_i, f_i$ , and  $h \equiv 0$  are as in Theorem 1. Applying the a.e.-version of Lemma 4, we find the functions  $A_i: J^n \rightarrow \mathbf{C}$ ,  $B_i: J^n \rightarrow \mathbf{R}$  ( $i = 1, \dots, N$ ) satisfying (ii)–(v) of Lemma 4 with  $G = \mathbf{R}$  and  $Y = J$  and such that (8) holds for a.e.  $(x, y) \in \mathbf{R} \times J^n$ .

By (iv) of Lemma 4,  $B_i - B_j$  is not constant on any subinterval of  $J^n$  for every  $i \neq j$ . Therefore, by Lemma 9,  $f_n$  equals a locally Lipschitz function  $\tilde{f}_n$  almost everywhere.

By Fubini's theorem, there is a subset  $Y$  of  $J^n$  of full measure such that for every  $y \in Y$ , (8) holds for a.e.  $x \in \mathbf{R}$ . Since  $f_n = \tilde{f}_n$  a.e., it follows that, for every  $y \in Y$ , we have

$$(13) \quad \sum_{i=1}^N A_i(y) \tilde{f}_n(x + B_i(y)) = 0$$

for a.e.  $x$ . Then, by the continuity of the functions  $\tilde{f}_n$  and  $B_i$  we find that (13) holds for every  $x \in \mathbf{R}$  and  $y \in Y$ .

Let  $L$  denote the set of continuous functions  $f \in C(\mathbf{R})$  satisfying

$$(14) \quad \sum_{i=1}^N A_i(y) f(x + B_i(y)) = 0$$

for every  $(x, y) \in \mathbf{R} \times Y$ . Then  $L$  is a translation invariant closed subspace of  $C(\mathbf{R})$  and, by the argument above,  $\tilde{f}_n \in L$ . If  $f \in L$  then (14) holds for a.e.  $(x, y) \in \mathbf{R} \times J^n$  and thus, by Lemma 9,  $f$  is locally Lipschitz. That is, each element of  $L$  is locally Lipschitz. We claim that there exists a nonzero difference operator  $\Delta$  such that  $\Delta f = 0$  for every  $f \in L$ . In fact, if  $f \in L$  then we have  $\Delta(y)f = 0$  for every  $y \in Y$ , where  $\Delta(y) = \sum_{i=1}^N A_i(y) T_{B_i(y)}$ . We have to show that  $\Delta(y)$  is nonzero for at least one  $y \in Y$ . But this is clear, because  $A_i(y) \neq 0$  for every  $y \in J^n$ , and  $B_1(y), \dots, B_n(y)$  are distinct on a dense open subset of  $J^n$ .

Therefore we may apply Lemma 3. We find that  $L$  is finite dimensional, and thus each element of  $L$  is an exponential polynomial. Since  $\tilde{f}_n \in L$  and  $f_n$  equals  $\tilde{f}_n$  almost everywhere, this completes the proof, assuming  $h \equiv 0$ .

The general case can be reduced to the previous one as follows. It is enough to show that  $f_n$  equals an exponential polynomial almost everywhere.

Let  $\Delta_b$  denote the difference operator defined by  $\Delta_b f(x) = f(x + b) - f(x)$ . Suppose that the functions  $a_i, b_i, f_i$ , and  $h$  are as in Theorem 1. Then we have

$$\sum_{i=1}^n a_i(y) \Delta_b f_i(x + b_i(y)) = 0$$

for almost every  $(x, y) \in \mathbf{R} \times J$  and for every  $b \in \mathbf{R}$ . As we proved already, this implies that  $\Delta_b f_n$  equals an exponential polynomial almost everywhere for every  $b \in \mathbf{R}$ . Then, in particular,  $\Delta_b f_n$  equals a continuous function almost everywhere for each  $b \in \mathbf{R}$ . By a theorem of T. Keleti [7, Theorem 2.9] it follows that  $f_n$  equals a continuous function  $\tilde{f}_n$  almost everywhere. Since  $\Delta_b \tilde{f}_n$  equals an exponential polynomial almost everywhere and  $\tilde{f}_n$  is continuous, we find that  $\Delta_b \tilde{f}_n$  equals an exponential polynomial everywhere for every  $b \in \mathbf{R}$ . Therefore, by a theorem of F. W. Carroll [2],  $\tilde{f}_n$  is exponential polynomial, which completes the proof.  $\square$

### 6. PROOF OF THEOREM 2

For every  $E \in \mathcal{E}$  we shall denote by  $\Lambda(E)$  the set of roots of  $E$ .

LEMMA 10. *Shapiro's conjecture implies that if  $\{E_j : j \in J\}$  is a system of exponential polynomials with constant coefficients such that  $\bigcap_{j \in J} \Lambda(E_j)$  is infinite, then either*

- (i) *there is a non-unit exponential polynomial that divides each  $E_j$ , or*
- (ii) *there is a nonzero complex number  $\gamma$  such that each  $E_j$  has a divisor of the form  $e^{r\gamma z} - c$ , where  $c \neq 0$  and  $r \neq 0$  is rational.*

*Proof.* By Ritt's theorem [9] we have  $E_j = F_j \cdot G_j$  ( $j \in J$ ), where each  $F_j$  is the product of finitely many simple exponential polynomials, and each  $G_j$  is the product of finitely many irreducible factors. Let  $\Lambda = \bigcap_{j \in J} \Lambda(E_j)$ . Then  $\Lambda \subset \Lambda(E_j) = \Lambda(F_j) \cup \Lambda(G_j)$  for every  $j \in J$ . Suppose that there exists a  $j_0 \in J$  such that  $\Lambda \cap \Lambda(G_{j_0})$  is infinite. Then there is an irreducible factor  $H$  of  $G_{j_0}$  such that  $\Lambda \cap \Lambda(H)$  is infinite. Then  $\Lambda(H) \cap \Lambda(E_j)$  is infinite for every  $i \in J$ , as it contains  $\Lambda \cap \Lambda(H)$ . If Shapiro's conjecture is true then  $H$  and  $E_j$  have a common non-unit factor. Since  $H$  is irreducible, this factor must be (a unit multiple of)  $H$ . Thus, in this case,  $H$  divides each  $E_j$ ; that is, (i) holds.

Next suppose that  $\Lambda \cap \Lambda(G_j)$  is finite for every  $j \in J$ . Then  $\Lambda \cap \Lambda(F_j)$  must be infinite for every  $j$ . It is easy to see that if  $F \in \mathcal{E}$  is simple then  $F$  is the product of a unit and of finitely many factors of the form  $e^{az} - c$ , where  $a \neq 0$  and  $c \neq 0$ . Therefore,  $\Lambda(F)$  is the union of finitely many arithmetical progressions (AP's).

Let  $j_0 \in J$  be arbitrary. Since  $\Lambda \cap \Lambda(F_{j_0})$  is infinite and  $\Lambda(F_{j_0})$  is the union of finitely many AP's, there exists an arithmetical progression  $A = \{b + nd : n \in \mathbf{Z}\}$  such that  $\Lambda \cap A$  is infinite. Let  $\gamma = d/(2\pi i)$ . We show that every  $E_i$  has a divisor of the form  $e^{r\gamma z} - c$ , where  $c \neq 0$  and  $r \neq 0$  is rational. That is, in this case (ii) holds.

Let  $j \in J$  be arbitrary. Since  $\Lambda(G_j) \cap (\Lambda \cap A)$  is finite, there is a factor  $e^{az} - c$  of  $F_j$  such that  $\Lambda(e^{az} - c) \cap (\Lambda \cap A)$  is infinite. Now  $\Lambda(e^{az} - c)$  is an AP with difference  $(2\pi i)/a$ , and thus  $(2\pi i)/a$  and  $d$  must be commensurable; that is,  $(2\pi i)/ad$  is rational. Thus  $a/\gamma = r$  is rational, which completes the proof.  $\square$

REMARK. As the following simple example shows, we cannot omit case (ii) from the statement of Lemma 10. Let  $G_n$  ( $n = 1, 2, \dots$ ) be a sequence of non-associate irreducible exponential sums such that  $\{1, \dots, n!\} \subset \Lambda(G_n)$  for every  $n$ . Let  $E_n = (e^{\frac{2\pi i}{n}z} - 1) \cdot G_n$  ( $n = 1, 2, \dots$ ). It is easy to check that  $\{n! : n = 1, 2, \dots\} \subset \Lambda(E_n)$  for every  $n$ , but the  $E_n$ 's do not have a common non-unit divisor.

Now we turn to the proof of Theorem 2. First we consider the case when the function  $h$  is identically zero. Suppose (1). Clearly, it is enough to show that  $f_n$  is an exponential polynomial. By Lemma 4, there are functions  $A_i: Y^n \rightarrow \mathbf{C}$  and  $B_i: Y^n \rightarrow \mathbf{R}$  ( $i = 1, \dots, N = n!$ ) such that (8) holds for every  $y = (y_1, \dots, y_n) \in Y^n$ ,  $A_i$  is nonvanishing and  $B_i$  is continuous on  $Y^n$  for every  $i$ . Also, it follows from (iv) of Lemma 4 that  $B_i - B_j$  is not constant on any nonempty open subset of  $Y^n$  for every  $1 \leq i < j \leq N$ . Consequently, there is a nonempty open set  $U \subset Y^n$  such that  $B_1(y), \dots, B_N(y)$  are distinct and of the same order for every  $y \in U$ . We may assume that  $B_1(y) < \dots < B_N(y)$  ( $y \in U$ ).

Let  $L$  denote the set of functions  $f \in C(\mathbf{R})$  satisfying

$$(15) \quad \sum_{i=1}^N A_i(y) f(x + B_i(y)) = 0$$

for every  $(x, y) \in \mathbf{R} \times Y^n$ . Then  $L$  is a translation invariant closed subspace

of  $C(\mathbf{R})$ , and  $f_n \in L$ . By Schwartz's theorem, it is enough to show that  $L$  is finite dimensional.

First we shall prove that the spectrum  $\text{sp}(L) = \{\lambda \in \mathbf{C} : e^{\lambda x} \in L\}$  is finite. Suppose  $\lambda \in \text{sp}(L)$ . Then  $\sum_{i=1}^N A_i(y) e^{\lambda B_i(y)} = 0$  for every  $y \in Y^n$ ; that is,  $\lambda$  is a root of the exponential sum  $E_y(z) = \sum_{i=1}^N A_i(y) e^{B_i(y)z}$  for every  $y \in Y^n$ . We prove, assuming Shapiro's conjecture, that the exponential sums  $E_y$  have only a finite number of common roots. Suppose this is not true. Then, by Lemma 10, one of the following two statements must be true:

- (i) there is a non-unit exponential polynomial that divides each  $E_y$ , or
- (ii) there is a nonzero complex number  $\gamma$  such that each  $E_y$  has a divisor of the form  $e^{r\gamma z} - c$ , where  $c \neq 0$  and  $r \neq 0$  is rational.

We show that each of these statements contradicts the condition that  $B_i - B_j$  is not constant on nonempty open sets.

Suppose (i), and let  $\sum_{i=1}^k \gamma_i e^{\delta_i z}$  be a non-unit exponential polynomial that divides each  $E_y$ . We may assume that  $k \geq 2$ ,  $\gamma_1, \dots, \gamma_k$  are nonzero,  $\delta_1, \dots, \delta_k$  are distinct, and that  $\delta_1 = 0$ . Then we have

$$(16) \quad E_y(z) = \sum_{i=1}^k \gamma_i e^{\delta_i z} \cdot \sum_{j=1}^{m(y)} a_j(y) e^{\beta_j(y)z}$$

for every  $y \in Y^n$ , where  $a_1(y), \dots, a_{m(y)}(y)$  are nonzero and  $\beta_1(y), \dots, \beta_{m(y)}(y)$  are distinct for every  $y$ . By a theorem of Ritt, there is a complex number  $\delta$  such that each of the numbers  $\delta_i - \delta$  ( $i = 1, \dots, k$ ) and  $\beta_j(y) + \delta$  ( $j = 1, \dots, m(y)$ ) is a linear combination of  $B_1(y), \dots, B_N(y)$  with rational coefficients. (See [9, p. 585] and [3, Lemma 2].) Since  $B_i(y)$  is real for every  $i$  and  $y$ , it follows that  $\delta_i - \delta$  and  $\beta_j(y) + \delta$  are also real for every  $i, j$  and  $y$ . Now  $\delta_1 = 0$  implies that  $\delta$  is real, and thus  $\delta_i$  and  $\beta_j(y)$  are real for every  $i = 1, \dots, k$ ,  $j = 1, \dots, m(y)$  and  $y \in Y^n$ . We may assume that  $0 = \delta_1 < \dots < \delta_k$ .

Let  $K(m) = \{y \in U : m(y) = m\}$  ( $m = 1, 2, \dots$ ). Then  $U = \bigcup_{m=1}^{\infty} K(m)$ . Since  $Y^n$  is a Baire space, it follows that  $K(m)$  is not nowhere dense for at least one  $m$ . Fix such an  $m$ , and partition  $K(m)$  into  $m!$  subsets according to the ordering of the numbers  $\beta_1(y), \dots, \beta_m(y)$ . Then at least one of these subsets is not nowhere dense. In other words, there exists a non-nowhere dense subset  $K$  of  $K(m)$  such that the ordering of the numbers  $\beta_1(y), \dots, \beta_m(y)$  is the same for every  $y \in K$ . We may assume that  $\beta_1(y) < \dots < \beta_m(y)$  ( $y \in K$ ). By (16) and  $\delta_1 = 0$  we have  $B_1(y) = \beta_1(y)$  for every  $y \in K$ .

Let  $J$  denote the set of those indices  $j \in \{1, \dots, m\}$  for which  $\beta_j - \beta_1$  is constant on a non-nowhere dense subset of  $K$ . Obviously,  $1 \in J$ . Let  $j_0$  be the largest element of  $J$ , and let  $K_0$  be a non-nowhere dense subset of  $K$  such that  $\beta_{j_0} - \beta_1$  is constant on  $K_0$ . Put  $K_i = \{y \in K_0 : \delta_k + \beta_{j_0}(y) = B_i(y)\}$  ( $i = 2, \dots, N$ ). If  $y \in K_i$  then

$$B_i(y) - B_1(y) = \delta_k + \beta_{j_0}(y) - \beta_1(y),$$

and thus  $B_i(y) - B_1(y)$  is constant on  $K_i$ . Therefore,  $K_i$  is nowhere dense for every  $i = 2, \dots, N$ . Consequently, the set  $\bigcup_{i=2}^N K_i$  is also nowhere dense, and  $K' = K_0 \setminus \bigcup_{i=2}^N K_i$  is not. Note that  $\delta_k + \beta_{j_0}(y) \neq B_i(y)$  for every  $y \in K'$  and  $i = 2, \dots, N$ .

Let  $y \in K'$ . The product on the right hand side of (15) contains the term  $\gamma_k a_{j_0}(y) e^{(\delta_k + \beta_{j_0}(y))z}$ . Now  $\delta_k + \beta_{j_0}(y) > \delta_1 + \beta_1(y) = B_1(y)$  and  $\delta_k + \beta_{j_0}(y) \neq B_i(y)$  for every  $i \geq 2$  by  $y \in K'$ . Thus  $\delta_k + \beta_{j_0}(y) \neq B_i(y)$  for every  $i$  and, consequently, this term must be cancelled out by other terms. That is, there are indices  $i(y) < k$  and  $j(y) > j_0$  such that  $\delta_k + \beta_{j_0}(y) = \delta_{i(y)} + \beta_{j(y)}$ . Now there must exist indices  $i < k$  and  $j > j_0$  and a non-nowhere dense subset  $K''$  of  $K'$  such that  $i(y) = i$  and  $j(y) = j$  for every  $y \in K''$ . Then

$$\beta_j(y) - \beta_1(y) = (\beta_{j_0}(y) - \beta_1(y)) + (\delta_k - \delta_i)$$

for every  $y \in K''$ . Now  $\beta_{j_0} - \beta_1$  is constant on  $K''$  (even on  $K_0$ ), and thus so is  $\beta_j - \beta_1$ . Therefore,  $j \in J$ . This, however, contradicts the fact that  $j_0$  was the maximal element of  $J$ . This contradiction proves the finiteness of  $\text{sp}(L)$  in the case when (i) holds.

Next assume (ii). Then there is a nonzero complex number  $\gamma$  such that

$$(17) \quad E_y(z) = (e^{r(y)\gamma z} - c(y)) \cdot \sum_{j=1}^{m(y)} a_j(y) e^{\beta_j(y)z}$$

for every  $y \in Y^n$ , where  $r(y) \neq 0$  is rational,  $c(y), a_1(y), \dots, a_{m(y)}(y)$  are nonzero and  $\beta_1(y), \dots, \beta_{m(y)}(y)$  are distinct for every  $y$ . We can prove, in the same way as in the case (i), that the numbers  $\gamma$  and  $\beta_1(y), \dots, \beta_{m(y)}$  are real for every  $y$ . Since  $Y^n$  is a Baire space, there is a nonzero rational number  $r$  and there is a positive integer  $m$  such that the set  $R = \{y \in U : r(y) = r, m(y) = m\}$  is not nowhere dense. Then there is a non-nowhere dense subset  $R_0$  of  $R$  such that the ordering of the numbers  $\beta_1(y), \dots, \beta_m(y)$  is the same for every  $y \in R_0$ . We may assume that  $\beta_1(y) < \dots < \beta_m(y)$  ( $y \in R_0$ ). From this point we can arrive at a contradiction in the same way as in the case of (i), using (17) instead of (16).

This proves that  $\text{sp}(L)$  is finite. If  $e^{\lambda_1 z}, \dots, e^{\lambda_s z}$  are the only exponential functions contained in  $L$ , then every exponential polynomial contained in  $L$  must be of the form  $\sum_{i=1}^s p_i(z) e^{\lambda_i z}$ , where  $p_1, \dots, p_s$  are polynomials. Since the set of all polynomials is dense in  $C(\mathbf{R})$  and  $L \neq C(\mathbf{R})$ , it follows that the degrees of  $p_1, \dots, p_s$  must be bounded. As the set of exponential polynomials is dense in  $L$ , we find that each element of  $L$  is an exponential polynomial, which completes the proof of Theorem 2 in the case when  $h \equiv 0$ .

The general case can be reduced to the previous one in the same way as in the proof of Theorem 1. Again, it is enough to show that  $f_n$  is an exponential polynomial. Since  $\Delta_b f_n$  satisfies the homogeneous version of (1), it follows that  $\Delta_b f_n$  is an exponential polynomial for every  $b$ . Therefore, by Carroll's theorem [2],  $f_n$  is also an exponential polynomial.  $\square$

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