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ON SINGULARITIES, "PERESTROIKAS" AND DIFFERENTIAL GEOMETRY OF SPACE CURVES

by Ricardo URIBE-VARGAS

ABSTRACT. We study the geometry of smooth curves in Euclidean 3-space in a neighbourhood of *flattenings* (points at which the osculating plane is stationary) and of *Darboux vertices* (points at which the instantaneous axis of rotation of the Frenet trihedral is stationary). Concerning these special points, we present local and global theorems and describe all bifurcations which may occur in generic 1-parameter families of smooth curves. The definition of a flattening directly generalises to curves in higher-dimensional Euclidean spaces while the definition of a Darboux vertex has two possible generalisations, called a *Darboux vertex* and a *twisting*. We present new results on flattenings, Darboux vertices and twistings for curves in Euclidean spaces of arbitrary dimension.

0. INTRODUCTION

We present new theorems on local and global differential geometry of curves in Euclidean spaces. We follow the kinematic interpretation of the Frenet trihedral, initiated by Darboux [8] and described in [10]. When a point moves along a curve in Euclidean space $\mathbf{R}^3$, its Frenet trihedral $(\mathbf{t}, \mathbf{n}, \mathbf{b})$, parallely translated to the origin, defines a rigid motion around the origin called *Frenet motion*. The instantaneous axis of rotation of Frenet motion — which we call the *Darboux axis* — is determined by the *Darboux vector*: $\tilde{\mathbf{d}} = \tau\mathbf{t} + \kappa\mathbf{b}$, where κ and τ are the curvature and the torsion of the curve, respectively.

The endpoints of the translated vectors of the Frenet trihedral $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ and of the *normalised Darboux vector* $\mathbf{d} = \tilde{\mathbf{d}}/\sqrt{\kappa^2 + \tau^2}$ describe four curves $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{D}$, on the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$, called the *tangent, normal, binormal* and *Darboux indicatrices* of that curve, respectively.

A *flattening* (*Darboux vertex*) of a smooth curve $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ is a point at which the osculating plane (respectively the Darboux axis) is stationary.

For both Darboux vertices and flattenings the radius of the osculating circle of the tangent indicatrix $\mathbf{T}$ *is critical (at flattenings it has the maximal value:* 1*).*

Let $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ be a smoothly immersed curve. A Darboux vertex of γ for which the radius of the osculating circle of the tangent indicatrix has a local maximum (minimum) will be called an *M-D-vertex* (*m-D-vertex*, respectively). Write $M(\gamma)$, $m(\gamma)$ and $F(\gamma)$ for the number of its M-D-vertices, m-D-vertices and flattenings, respectively. We have discovered and proved (see Theorem 3) a universal relation between the number of points at which the Darboux axis is stationary (M-D-vertices and m-D-vertices) and the number of points at which the osculating plane is stationary (flattenings):

Any generic closed curve $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ *satisfies* $m(\gamma) - M(\gamma) - F(\gamma) = 0$.

A generic curve in Euclidean 3-space has no *inflection* (point where the curvature vanishes). However, a generic 1-parameter family of curves can have, at isolated parameter values, a curve having one isolated inflection. The smooth curves having an inflection form a *discriminant hypersurface* in the space of smooth curves. We study the local bifurcations of the curve when a generic 1-parameter family traverses this discriminant hypersurface. Given a curve, put the following labels to its special points: $\mathfrak{m}$ to the m-D-vertices, $\mathfrak{F}$ to the flattenings and $\mathfrak{I}$ to the inflections. Below, the symbol $\mathfrak{m}\mathfrak{F}\mathfrak{m}\mathfrak{F}\mathfrak{m}$ means the curve has exactly 5 special points arranged in that order — in a very small interval. For the bifurcations, the symbol $\longleftrightarrow$ represents the transition between two 'local situations' of the curve. Given a generic 1-parameter family of curves γ_t, suppose that for the parameter value of the family $t = 0$ the corresponding curve γ_0 has an inflection point (Theorem 1):

During an "inflection" perestroika, the curve experiences the following transition in a neighbourhood of the inflection point (see Figure 1):

$$\underset{t<0}{\mathfrak{m}} \quad\longleftrightarrow\quad \underset{t=0}{\mathfrak{I}} \quad\longleftrightarrow\quad \underset{t>0}{\mathfrak{m}\mathfrak{F}\mathfrak{m}\mathfrak{F}\mathfrak{m}}\,.$$

This theorem shows that there is a very rich geometry associated with inflections. Part of such a geometry appears in the transitions experienced by the tangent, normal, binormal and Darboux indicatrices, during an inflection perestroika. See Figures 4, 2, 5 and 3.

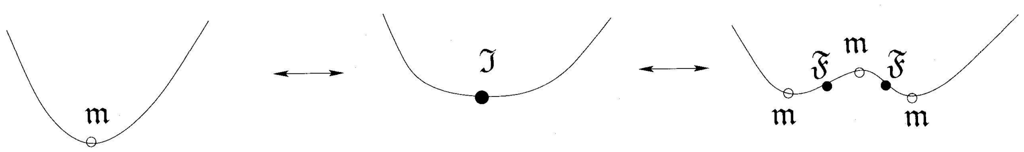

FIGURE 1
Schematic representation of the local transitions of a curve during an inflection Perestroika

Besides the inflection perestroika, there is only one other perestroika in generic 1-parameter families, the *biflattening perestroika*, for which the number of flattenings changes (Theorem 2). In both, Darboux vertices play an essential role: 2 flattenings can appear or disappear only at the Darboux vertices of the curve. So Darboux vertices are necessary to have such perestroikas. It is surprising that Darboux vertices had not been deeply studied before.

We can consider the tangent and binormal indicatrices as wave fronts on the unit sphere, while the Darboux indicatrix may be considered as a caustic. The flattenings and Darboux vertices of a curve in Euclidean 3-space correspond to semi-cubic cusps of these wave fronts and caustics, respectively. Such considerations were the starting point for the discovery and for the initial proofs of the results presented here for curves in $\mathbf{R}^3$. These results belong thus to the theory of singularities in symplectic and contact geometry (see [5], [3] and [16]). However, the perestroikas of wave fronts and caustics occurring during the inflection perestroika are not in the list of standard perestroikas (see [1] or [3]) of one-dimensional fronts and caustics!

When the Frenet frame of a curve in Euclidean space is translated to the origin, it determines a rigid motion which has an instantaneous axis of rotation (called *Darboux axis*) only if the space is of odd dimension. A point for which the Darboux axis is stationary is called a *Darboux vertex* of that curve (see Theorem 7):

Let γ be a smoothly immersed curve in $\mathbf{R}^{2k+1}$, $k \geq 1$. Write $\kappa_1, \kappa_2, \dots, \kappa_{2k}$ for its curvatures. The curve γ has a Darboux vertex at $s = s_0$ if and only if $(\kappa_2/\kappa_1)' = 0$, $(\kappa_4/\kappa_3)' = 0, \dots$, $(\kappa_{2k}/\kappa_{2k-1})' = 0$ at $s = s_0$.

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CONTENTS

1. PREPARATORY DEFINITIONS

DEFINITION. Let M be a d-dimensional submanifold of $\mathbf{R}^n$, considered as a complete intersection: $M = \{x \in \mathbf{R}^n : g_1(x) = \cdots = g_{n-d}(x) = 0\}$. We say that k is the *order of contact* of a curve $\gamma\colon t \mapsto \gamma(t) \in \mathbf{R}^n$ with the submanifold M, or that γ and M have *k-point contact*, at a point $\gamma(t_0)$, if at $t = t_0$ each function $g_1 \circ \gamma, \ldots, g_{n-d} \circ \gamma$ has a zero of multiplicity at least k and at least one of them has a zero of multiplicity k.

REMARK. To make this definition more invariant, one could denote the image of γ by Γ and then write that the *order of contact* at a point is the minimum of the multiplicities of zero among the functions of the form $g|_\Gamma\colon \Gamma \to \mathbf{R}$, at that point, where $g\colon \mathbf{R}^n \to \mathbf{R}$ belongs to the generating ideal of M and we assume that 0 is a regular value of g.

EXAMPLE. A smooth curve in $\mathbf{R}^n$ has 2-point contact with its tangent line (at the point of tangency) for the generic points of the curve. The curve $y = x^3$ has 3-point contact with the line $y = 0$, at the origin: the equation $x^3 = 0$ has a root of multiplicity 3.

DEFINITION. An *osculating hyperplane* at a point of a curve in Euclidean (or affine or projective) n-space is a hyperplane having at least n-point contact with the curve at that point.

REMARK. At any point of a generic curve the osculating hyperplane is unique and it is spanned by the first $n-1$ derivatives of the curve at that point, which are linearly independent.

DEFINITION. A *flattening* of a smoothly immersed curve in $\mathbf{R}^n$ is a point at which the curve has at least $(n+1)$-point contact with its osculating hyperplane. Equivalently, a flattening is a point at which the osculating hyperplane is stationary. Or equivalently, a flattening is a point for which the n-th derivative of the curve lies on the osculating hyperplane.

EXAMPLE. The flattenings of a curve in $\mathbf{R}^3$ with non-zero curvature are the points at which the torsion vanishes.

PART I. CURVES IN EUCLIDEAN 3-SPACE

2. PRELIMINARY RESULTS FOR CURVES IN $\mathbf{R}^3$

2.1 THE "AMBIGUITY" OF THE FRENET TRIHEDRAL

We have introduced the notion of Darboux indicatrix and Darboux vector of a curve $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ using the kinematic interpretation of the Frenet trihedral. However we must pay attention to the following

MAIN REMARK. The Frenet trihedral is well defined for curves with positive curvature everywhere (a generic curve has positive curvature everywhere). However, a difficulty occurs when the curvature is vanishing somewhere (that is, at the inflections); in this case there is some ambiguity in the definitions of Frenet trihedral and of curvature. One needs to make some choices. The usual conventions are:

(a) one supposes that $\kappa \geq 0$ and takes the normal $\mathbf{n}$ accordingly;

(b) one supposes that $\kappa(s)$ and $\mathbf{n}(s)$ are smooth functions of s (Fenchel, [10]).

None of these choices takes into account that at an inflection all planes containing the tangent line to the curve are osculating. Moreover, when one considers not a fixed curve but a family of curves, none of these choices is consistent. Our choice takes these facts into account:

(c) We suppose that $\kappa \geq 0$. For $\kappa > 0$ we take the direction of the normal $\mathbf{n}$ accordingly; but we suppose that at the inflections the Frenet trihedral is not unique: the vector $\mathbf{t}$ together with each unit vector $\mathbf{v}$ normal to γ define a Frenet trihedral: $\mathbf{t}$, $\mathbf{n} = \mathbf{v}$, $\mathbf{b} = \mathbf{t} \times \mathbf{v}$.

In Figures 5 and 3, to the inflection of a space curve there corresponds a whole great circle in its normal and in its binormal indicatrices, respectively.

REMARK. When the curve has an inflection the trihedral's Frenet motion is not continuous.

2.2 FENCHEL'S STATEMENTS

Let γ be an immersed curve in $\mathbf{R}^3$ and let $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ and $\mathbf{D}$ its tangent, normal, binormal and Darboux indicatrices, respectively. The following facts are well known ([10]):

(1) *The indicatrices* $\mathbf{D}$ *and* $-\mathbf{D}$ *form the spherical caustic of* $\mathbf{T}$ *(and of* $\mathbf{B}$*) that is,* $\mathbf{D}$ *and* $-\mathbf{D}$ *form the envelope of the family of great circles of* $\mathbf{S}^2$ *orthogonal to* $\mathbf{T}$ *(they are also orthogonal to* $\mathbf{B}$*).*

(2) *To a spherical inflection of* $\mathbf{T}$ *there corresponds a cusp of* $\mathbf{B}$ *and vice versa.*

(3) *An inflection of* γ *corresponds to a cusp of* $\mathbf{T}$ *and to a spherical inflection of* $\mathbf{B}$.

(4) *The inflections of* $\mathbf{N}$ *and the cusps of* $\mathbf{D}$ *correspond to the points at which* τ/κ *is stationary.*

REMARK. The 'vice versa' of item (2) and the statement about $\mathbf{B}$ of item (3) hold for a fixed curve and only under Fenchel's convention (b) of the Main Remark above. The author discovered all these facts by applying the theory of wave fronts on the sphere $\mathbf{S}^2$, developed by Arnold [5], to the curve $\mathbf{T}$.

2.3 MAIN LEMMAS ON DARBOUX VERTICES AND FLATTENINGS

Consider the parameter ϑ of a smoothly immersed curve $\gamma\colon \vartheta \mapsto \gamma(\vartheta) \in \mathbf{R}^3$ to be the time.

LEMMA 2.1. *The Darboux axis at time* ϑ *is determined by the kernel of the Frenet matrix (given with respect to the basis* $\mathbf{t}, \mathbf{n}, \mathbf{b}$*):*

$$M(\vartheta) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix},$$

where κ *is the curvature and* τ *the torsion of* γ *at time* ϑ.

Proof. This is a direct consequence of the Frenet equations. □

REMARK. Evidently, the Darboux vector $\tilde{\mathbf{d}} = \kappa\mathbf{b} + \tau\mathbf{t}$ lies in the kernel of M. We assume that κ and τ never vanish at the same point.

REMARK. The point $\gamma(\vartheta)$ is a flattening of γ if and only if the Darboux vector of γ at ϑ is proportional to the binormal vector of γ at ϑ.

The following proposition was proved in [9] by means of geometric considerations.

LEMMA 2.2. *The geodesic curvature of the tangent indicatrix* $\mathbf{T} \subset \mathbf{S}^2$ *of a curve* γ *with curvature* $\kappa \neq 0$ *and torsion* τ *is equal to the function* τ/κ.

Proof. Let s and r be the arc length parameters of γ and of its tangent indicatrix, respectively. So $d\mathbf{T}/dr = \mathbf{n}$ and $ds/dr = 1/\kappa$. The geodesic curvature of $\mathbf{T}$ is given by the orthogonal projection of $d\mathbf{n}/dr$ to the plane orthogonal to $\mathbf{t}$. We have that $d\mathbf{n}/dr = (d\mathbf{n}/ds)(ds/dr) = (-\kappa\mathbf{t} + \tau\mathbf{b})(1/\kappa)$. □

REMARK. At a Darboux vertex of a curve in $\mathbf{R}^3$ the first derivative of the Darboux indicatrix vanishes (i.e. it is a semi-cubic cusp of the Darboux indicatrix if the point is a generic Darboux vertex).

LEMMA 2.3. *The Darboux vertices of a curve* γ *are the critical points of the geodesic curvature* (τ/κ) *of its tangent indicatrix* $(\kappa \neq 0)$.

Proof. We write $\omega = (\kappa^2 + \tau^2)^{1/2}$. So $\mathbf{D} = (\kappa/\omega)\mathbf{b} + (\tau/\omega)\mathbf{t}$ and the derivative of $\mathbf{D}$ with respect to the arc length of γ is $\mathbf{D}' = \frac{(\kappa\tau' - \tau\kappa')}{\omega^3}(\kappa\mathbf{t} - \tau\mathbf{b})$. So $\mathbf{D}' = 0$ if and only if $(\tau/\kappa)' = 0$. □

LEMMA 2.4. *The flattenings of a curve* γ *in* $\mathbf{R}^3$ *correspond to the spherical inflections of its tangent indicatrix* $\mathbf{T}$.

Proof. At a spherical inflection of $\mathbf{T}$ the second derivative of $\mathbf{T}$ belongs to the plane generated by $\mathbf{T}$ and $d\mathbf{T}/ds$. So the first three derivatives of γ are linearly dependent. Thus this corresponds to a flattening of γ. □

3. Perestroikas of curves in $\mathbf{R}^3$

We describe how Darboux vertices behave during the perestroikas in generic one-parameter families of curves. We describe the corresponding perestroikas of the Darboux indicatrix, of the tangent indicatrix and of the bitangent indicatrix.

Definition. A point p of a curve in $\mathbf{R}^n$ (in $\mathbf{R}P^n$) is of *type* $(a_1, \dots, a_n)$ if, in suitable affine coordinates centered at p, the curve is the image of the smooth mapping $x_1 = t^{a_1} + o(t^{a_1}), \dots, x_n = t^{a_n} + o(t^{a_n})$, where $a_1 < \dots < a_n$ are the smallest possible natural numbers for the representation.

Example [14]. An ordinary point of a curve in $\mathbf{R}^3$ (in $\mathbf{R}P^3$) is a point of type $(1,2,3)$, the simplest flattening is of type $(1,2,4)$, the biflattening is of type $(1,2,5)$ and the simplest inflection is of type $(1,3,4)$.

Remark [14]. *Besides the ordinary points, a generic curve in* $\mathbf{R}^3$ *(in* $\mathbf{R}P^3$*) can only have isolated flattenings. Besides the generic curves, a generic one-parameter family of smooth curves in* $\mathbf{R}^3$ *can only have isolated parameter values for which the corresponding curve has one inflection or one biflattening. When the parameter of the family passes through one of such isolated values, the number of flattenings of the corresponding curve changes by* 2.

Let γ be an immersed curve in $\mathbf{R}^3$ and $\mathbf{T}$ be its tangent indicatrix. At the simple Darboux vertices of γ the order of contact of $\mathbf{T}$ with its osculating circle is 4 (at the ordinary points it is 3), and the radius of the osculating circle of $\mathbf{T}$ is a local maximum or a local minimum (Lemma 2.3). The points of spherical inflection of $\mathbf{T}$ (the flattenings of γ) are not Darboux vertices (the order of contact with the osculating circle is 3), but the radius of the osculating circle has the maximal value: 1. The radius of the osculating circle of $\mathbf{T}$ at a Darboux vertex of γ can be 1 only at a spherical bi-inflection of $\mathbf{T}$ (a biflattening of γ), thus such a Darboux vertex is not generic.

A generic Darboux vertex for which the radius of the osculating circle of the tangent indicatrix has a local maximum (minimum) will be called an *M-D-vertex* (*m-D-vertex*, respectively).

Given a curve, put the following labels to its special points: $\mathfrak{m}$ to the m-D-vertices, $\mathfrak{M}$ to the M-D-vertices, $\mathfrak{F}$ to the flattenings, $\mathfrak{B}$ to the biflattenings and $\mathfrak{I}$ to the inflections.

THEOREM 1. *During an inflection perestroika, the curve experiences the following transition in a neighbourhood of the inflection:*

$$\mathfrak{m} \longleftrightarrow \mathfrak{I} \longleftrightarrow \mathfrak{m}\mathfrak{F}\mathfrak{m}\mathfrak{F}\mathfrak{m}\,.$$

THEOREM 2. *During a biflattening perestroika, the curve experiences the following transition in a neighbourhood of the biflattening point:*

$$\mathfrak{M} \longleftrightarrow \mathfrak{B} \longleftrightarrow \mathfrak{F}\mathfrak{m}\mathfrak{F}\,.$$

REMARK. Besides the perestroikas of Theorems 1 and 2, there is only one other perestroika (the *double Darboux vertex perestroika*) at which the number of Darboux vertices changes: two Darboux vertices are born or killed.

3.1 PROOF OF THEOREM 2

Let γ_t be a generic one-parameter family of curves having a biflattening perestroika at $t = 0$. Suppose that at the perestroika two flattenings are born. This means — by Lemma 2.4 — that two spherical inflections of the tangent indicatrix $\mathbf{T}_t$ are born by means of a spherical bi-inflection transition. Thus before the perestroika the radius of the osculating circle of $\mathbf{T}_t$ has a local maximum: an M-D-vertex. At the biflattening moment, this radius will take the value 1. After the perestroika two close spherical inflections of $\mathbf{T}_t$ are born. The radius of the osculating circle of both is 1. So in the small segment of curve between them there is a point of $\mathbf{T}_t$ for which the radius of the osculating circle has a local minimum, i.e., an m-D-vertex of γ_t. □

3.2 PROOF OF THEOREM 1

Let γ_t be a generic one-parameter family of curves having an inflection perestroika at $t = 0$. Suppose that at the perestroika two flattenings are born. This means — by Lemma 2.4 — that two spherical inflections of the tangent indicatrix $\mathbf{T}_t$ are born by means of a cusp transition (see Figure 2).

So, before the perestroika ($t < 0$) there is an m-D-vertex $\mathfrak{m}_t$ for which the radius of the osculating circle of $\mathbf{T}_t$ is very small and is going to zero (at the cusp moment). Thus going along the curve γ_t from the point $\mathfrak{m}_t$, on each side of it, the radius of the osculating circle of $\mathbf{T}_t$ is increasing. After the perestroika ($t > 0$) we have two close spherical inflections of $\mathbf{T}_t$. In the short segment of curve between them there is a point of $\mathbf{T}_t$ having a local minimum of the radius of the osculating circle: an m-D-vertex of γ_t.

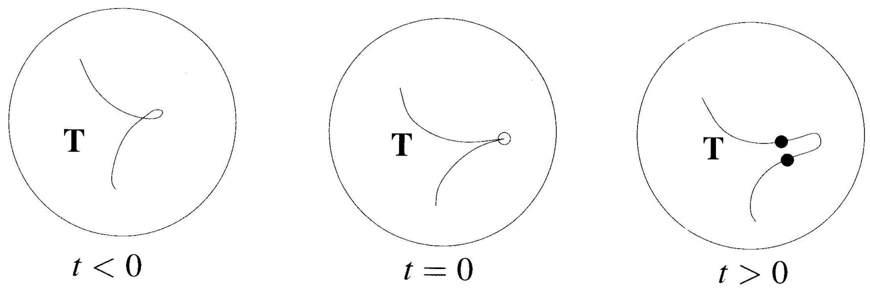

FIGURE 2
Transition of the tangent indicatrix during an inflection perestroika

On the other side of these spherical inflections the radius of the osculating circle of $\mathbf{T}_t$ is decreasing. But before the perestroika (and also at the moment of perestroika) the radius of the osculating circle was increasing going from the point $\mathfrak{m}_t$ along the two branches of the curve $\mathbf{T}_t$ (locally separated by $\mathfrak{m}_t$). The fact that the perestroika is a local transition implies that near each of these spherical inflections and outside the small segment bounded by them, there is also a local minimum of the radius of the osculating circle. Thus besides the m-D-vertex located between the two new flattenings of γ_t $(t > 0)$ we have two new m-D-vertices located outside the small segment of curve bounded by these flattenings. $\square$

4. GLOBAL THEOREMS

Let γ be a smooth closed curve in $\mathbf{R}^3$. Write $M(\gamma)$, $m(\gamma)$ and $F(\gamma)$ for the number of its M-D-vertices, m-D-vertices and flattenings, respectively. The number of points at which the Darboux axis or the osculating plane are stationary satisfies a universal relation:

THEOREM 3. *Any generic closed curve* $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ *satisfies*

$$m(\gamma) - M(\gamma) - F(\gamma) = 0\,.$$

There is a version of Theorem 3 for non-closed curves $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ whose tangent indicatrix $\mathbf{T}_\gamma$ is smooth and periodic (closed):

THEOREM $\tilde{3}$. *Let $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ be a smooth curve without inflections. Suppose that the image of its tangent indicatrix $\mathbf{T}_\gamma\colon \mathbf{R} \to \mathbf{S}^2 \subset \mathbf{R}^3$ is a closed smoothly immersed curve Γ (which may be covered an infinite number of times). Let $I_a^b = [a,b) \subset \mathbf{R}$ be any interval such that $\Gamma = \mathbf{T}_\gamma(I_a^b)$ and each point of Γ has exactly one preimage. Then the curve $\gamma_{ab} = \gamma|_{I_a^b} \to \mathbf{R}^3$ satisfies*

$$m(\gamma_{ab}) - M(\gamma_{ab}) - F(\gamma_{ab}) = 0\,.$$

THEOREM 4. *If a generic closed curve has an M-D-vertex then it also has two different m-D-vertices such that the open segment of the curve bounded by them and containing that M-D-vertex has no flattening and has no other Darboux vertex.*

Proof. Consider a generic smooth immersion $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ and write $F(\gamma)$ and $D(\gamma)$ for the number of its flattenings and Darboux vertices, respectively. In [11] Heil proved a theorem implying that *for a generic closed curve (i.e. without inflections and possibly with simple flattenings and simple Darboux vertices) γ we have $F(\gamma) + D(\gamma) \geq 4$.*

Suppose that γ has an M-D-vertex and denote it by M. By Heil's theorem, besides M there are at least three other Darboux vertices and/or flattenings. The radius of the osculating circle of $\mathbf{T}$ decreases when you move along the curve from M. Thus M must have two neighbouring m-D-vertices. □

4.1 PROOF OF THEOREMS 3 AND $\tilde{3}$

First proof. Let γ_t be a generic family of closed curves of $\mathbf{R}^3$. Besides the perestroikas of generic one-parameter families of curves at which the number of flattenings changes (the perestroikas of Theorems 1 and 2), there is other perestroika — called *double Darboux vertex* perestroika — at which two Darboux vertices (one m-D-vertex and one M-D-vertex) are born or killed. At all of them the number $m(\gamma_t) - M(\gamma_t) - F(\gamma_t)$ does not change. The reader can verify that $m(\gamma) - M(\gamma) - F(\gamma) = 0$ for a generic example. □

Second proof. For a generic curve γ, the local extrema of the radius of the osculating circle of $\mathbf{T}$ must alternate. If $\mathbf{T}$ is a smooth closed curve the number of local minima is equal to the number of local maxima of the radius of the osculating circle of $\mathbf{T}$. The local minima correspond to the m-D-vertices of γ. The local maxima correspond to the M-D-vertices or to the flattenings of γ. Thus $m(\gamma) - M(\gamma) - F(\gamma) = 0$. □

5. THE TRANSITIONS OF THE SPHERICAL INDICATRICES DURING AN INFLECTION PERESTROIKA

During an inflection perestroika, the normal **N** (and $-\mathbf{N}$), binormal **B** (and $-\mathbf{B}$) and Darboux **D** (and $-\mathbf{D}$) indicatrices experience global changes.

TRANSITION OF THE BINORMAL INDICATRIX. In Figure 3 we describe the perestroika of the family of binormal indicatrices corresponding to a generic one-parameter family of curves γ_t in $\mathbf{R}^3$, during an inflection perestroika. In the same figure, the perestroika of the corresponding family of tangent indicatrices is described by the small curve on the left hand side of each sphere. The binormal indicatrix is obtained from the tangent indicatrix by a translation along the great circles normal to the tangent indicatrix at a distance of $\pm\pi/2$. Note that at the inflection moment, a whole great circle in the binormal indicatrix corresponds to the point of inflection, i.e. to the cusp of the tangent indicatrix (see Main Remark in §2.1). The local perestroika of the space curve γ involves a global perestroika of $\pm$ the binormal indicatrix which "includes" not only the neighbourhood of the cusp points but also this whole great circle. In particular, before and after the perestroika the two curves **B** and $-\mathbf{B}$ have a natural orientation given by the orientation of the original curve γ. The orientation of each arc of this great circle (not containing the bifurcation cusp points) changes at the inflection moment.

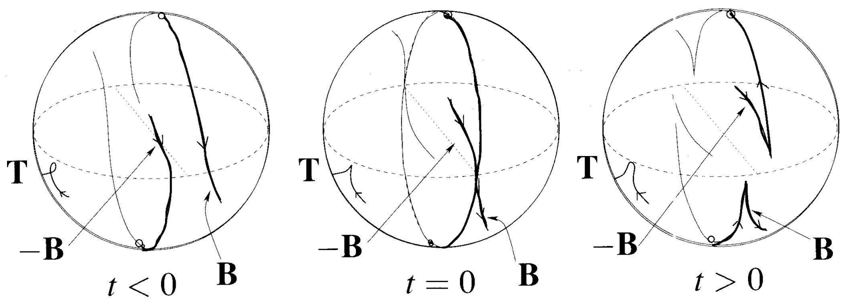

FIGURE 3
Perestroika of $\pm$ the binormal indicatrix during an inflection perestroika

Note that for $t<0$ and $t>0$, the North and South poles of the sphere in $\mathbf{B}_t$ and $-\mathbf{B}_t$, respectively correspond to the point of maximal curvature of the tangent indicatrix $\mathbf{T}_t$. This means that the image of the point of minimal curvature of the curve γ_t on the binormal indicatrix (that is, the point which "will become" the inflection of γ_0) is far from the bifurcation cusp points of $\mathbf{B}_0$ and $-\mathbf{B}_0$. As is shown in Figure 3, the curves $\mathbf{B}$ and $-\mathbf{B}$ "exchange" a component (each one "gives" a half of great circle to the other).

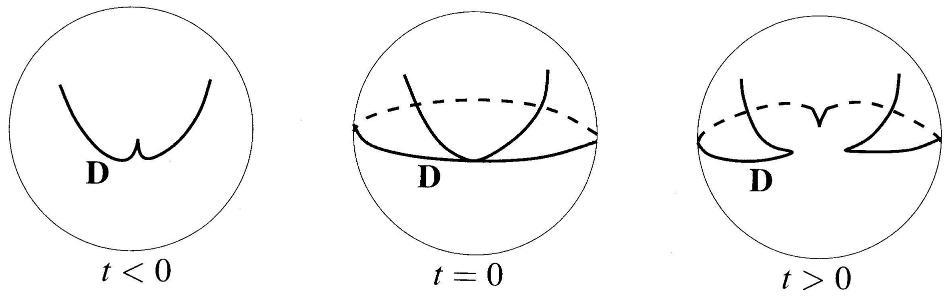

FIGURE 4
Darboux indicatrix during an inflection perestroika

TRANSITION OF THE DARBOUX INDICATRIX. For all parameter values, except the moment of inflection perestroika, the Darboux indicatrix has cusps (1 or 3) corresponding to Darboux vertices of the curve $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ (Theorem 1). At the moment of perestroika these cusps disappear and the Darboux indicatrix $\mathbf{D}$ (and $-\mathbf{D}$) "becomes smooth" but a new component (a great circle) is added to $\mathbf{D}$ (and also to $-\mathbf{D}$) see Figure 4.

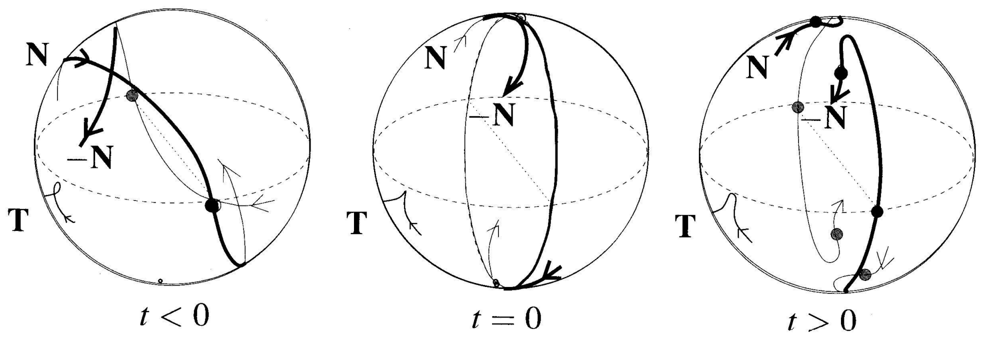

FIGURE 5
Perestroika of the normal indicatrix during an inflection perestroika

TRANSITION OF THE NORMAL INDICATRIX. In Figure 5 we describe the perestroika of the normal indicatrices of a generic one-parameter family of curves γ_t in $\mathbf{R}^3$ during an inflection perestroika. We also describe the corresponding perestroika of the tangent indicatrices: it is the small curve on the left hand side of each sphere. The normal indicatrix $\mathbf{N}$ and its antipodal curve $-\mathbf{N}$, are obtained from the tangent indicatrix by a translation at a distance $\pm\pi/2$ along the great circles tangent to the tangent indicatrix. At the inflection moment, a whole great circle in the union of the curves $\mathbf{N}$ and $-\mathbf{N}$ is associated to the point of inflection (i.e. to the cusp of the tangent indicatrix). The points of spherical inflection of $\mathbf{N}$ or $-\mathbf{N}$ at time t correspond to the Darboux vertices of the curve γ_t.

6. Special points and perestroikas from the contact and symplectic view points

THE SPACE OF TRIHEDRALS. The space of orthonormal-oriented bases of $\mathbf{R}^3$ (trihedrals) is isomorphic to the contact 3-dimensional manifold $PT^*\mathbf{S}^2$ of co-oriented contact elements on the sphere $\mathbf{S}^2$. When a point moves along a curve γ in $\mathbf{R}^3$, *the Frenet trihedral of the curve at that moving point sweeps a Legendrian curve* $L \in PT^*\mathbf{S}^2$. The three natural projections $L \to \mathbf{S}^2$ of this Legendre curve to the unit sphere — defined by the choice of the first, the second or the third vector of the Frenet trihedral — give the tangent, normal and binormal indicatrices of γ, respectively.

Figures 2, 3 and 5 correspond to the three natural projections to the sphere of one and the same family L_t of Legendrian curves in $PT^*\mathbf{S}^2$ associated to a family γ_t of curves in $\mathbf{R}^3$ during an inflection perestroika (see [16] and "Triality Theorem" in [5]). In this case the Legendrian curve L_t experiences a perestroika at the inflection moment (see Figure 6). This explains why these non-standard perestroikas of caustics and wave fronts appear naturally during an inflection perestroika of a smooth curve.

THE GAUSS MAP. We recall from the theory of Lagrangian singularities (see for instance [2] or [3]) that the Gauss map of a smooth curve in Euclidean space $\mathbf{R}^3$ associates to each oriented line normal to the curve its orienting unit vector, translated to the origin. This defines a (Lagrangian) map of the (Lagrangian) cylinder, formed by all the oriented lines normal to the curve, into the unit sphere $\mathbf{S}^2$. We recall that the *caustic of a Lagrangian map* is the set of its critical values. The images of all oriented lines normal to the curve whose orienting unit vector is plus or minus the binormal vector, $\pm\mathbf{b}$, form the caustic of the Gauss map. In other words the caustic of the Gauss

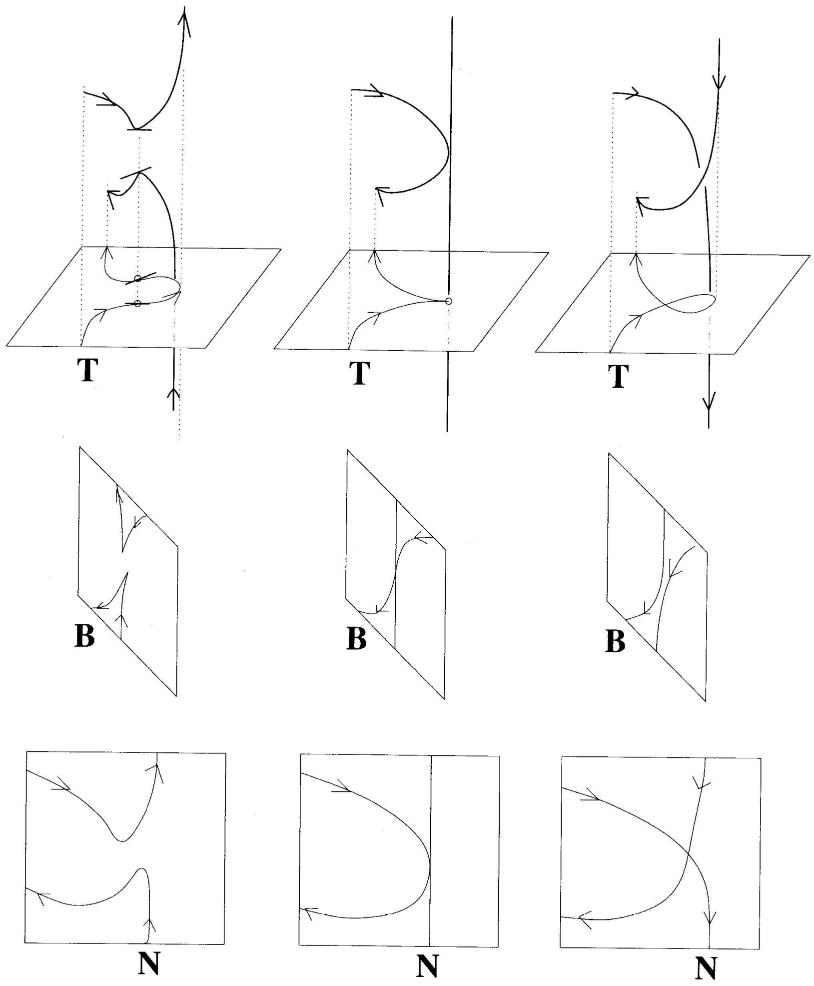

FIGURE 6
The three natural projections of the Legendrian curve (of Frenet trihedrals) associated to a space curve

map consists of the binormal indicatrix $\mathbf{B}$ and its antipodal curve $-\mathbf{B}$. At the flattenings of the curve each component of the caustic has a cusp.

EXAMPLE (4-FLATTENING THEOREM). Given a closed convex plane curve γ in $\mathbf{R}^2 \subset \mathbf{R}^3$, the binormal vector is constant. Thus the caustic of the Gauss map consists of the North and South poles of the unit sphere, which are the images of $\mathbf{b}$ and $-\mathbf{b}$, respectively. *Any small enough generic perturbation* $\widehat{\gamma}$ *of* γ *in* $\mathbf{R}^3$ *(taking the derivatives into account) has at least* 4 *flattenings* [4]. In terms of Lagrangian singularities the theorem says that *the caustic of the Gauss map associated to* $\widehat{\gamma}$ *(i.e.* $\pm$ *the binormal indicatrix of* $\widehat{\gamma}$*) consists of two small antipodal curves near the poles each one having at least* 4 *cusps.*

REMARK. Consider a generic one-parameter family γ_t of smoothly immersed curves in $\mathbf{R}^3$ having a curve with an inflection for an isolated value $t=0$ of the parameter. The Gauss map associated to each curve of the family has a caustic. All these caustics form a one-parameter family $\mathbf{B}_t$ of caustics having a non-standard perestroika at $t=0$ (see Figure 3).

REMARK. During the biflattening perestroika of Theorem 3 the Darboux indicatrix does not change its shape (it has a semi-cubic cusp for all neighbouring parameter values). During a biflattening perestroika the binormal indicatrix experiences a typical local perestroika of caustics, see Figure 7.

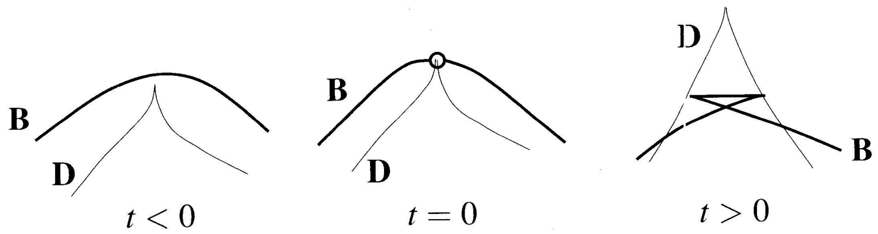

FIGURE 7
Darboux and binormal indicatrices during a biflattening perestroika

During a double Darboux vertex perestroika, the Darboux indicatrix experiences a typical local perestroika of caustics, see Figure 8.

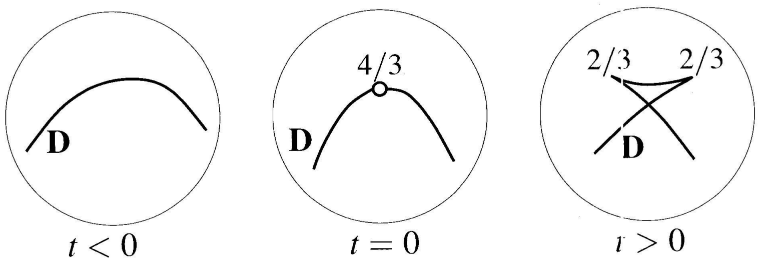

FIGURE 8
Darboux indicatrix during a double Darboux vertex perestroika

The *front* of the tangential map of a generic curve γ in $\mathbf{R}^3$ (in $\mathbf{R}P^3$) is a surface in the dual space $(\mathbf{R}^3)^\vee$ (respect. $(\mathbf{R}P^3)^\vee$) consisting of all planes of $\mathbf{R}^3$ (respect. of $\mathbf{R}P^3$) tangent to γ. The perestroika of the family of fronts of a generic one-parameter family of curves in $\mathbf{R}^3$ (respect. $\mathbf{R}P^3$) having a curve with an inflection for an isolated parameter value is described in [14].

7. Special points and perestroikas on the angle-length strip

Let γ be a smooth immersed curve in $\mathbf{R}^3$, parametrised by the arc length s. The *rectifying plane* of the curve at s is the plane generated by the unit tangent and binormal vectors of γ at s. As we saw in §2, the normalised Darboux vector lies in the rectifying plane: $\mathbf{d} = (\tau/\omega)\mathbf{t} + (\kappa/\omega)\mathbf{b}$. Hence we can write $\tau/\omega = \sin\vartheta$ and $\kappa/\omega = \cos\vartheta$, where ϑ is the signed angle from $\mathbf{b}$ to $\mathbf{d}$, where the ordered pair $(\mathbf{b}, \mathbf{t})$ is a positive basis of the rectifying plane.

The formula $\mathbf{D}' = \frac{(\kappa\tau' - \tau\kappa')}{\omega^3}(\kappa\mathbf{t} - \tau\mathbf{b})$ implies that the derivative of the Darboux indicatrix also lies in the rectifying plane (proving also Fenchel's statement 1).

Any generic curve satisfies the condition $\kappa > 0$. This fact implies that $\mathbf{D}$ lies in the half-plane of the rectifying plane given by the direction of $\mathbf{b}$ (see Figure 9).

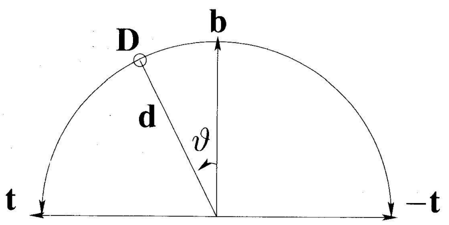

FIGURE 9
The Darboux indicatrix in the rectifying plane

The Darboux indicatrix in fact lies in a semi-circle, and $\mathbf{D}$ can be identified with the angle ϑ, where $\vartheta \in [-\pi/2, \pi/2]$ (and $\vartheta = \pm\pi/2$ only at inflections). We will describe how the Darboux indicatrix of a generic family of curves evolves. We will schematically represent the curve $\mathbf{D}$ as a curve $\widetilde{\mathbf{D}} = (\vartheta(s), s)$ in the angle-length strip $[-\pi/2, \pi/2] \times \mathbf{R}$.

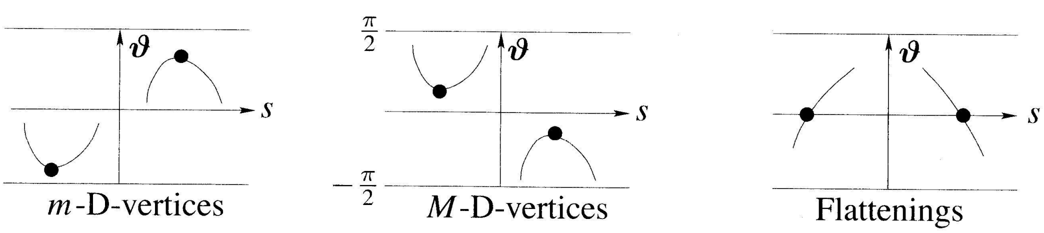

FIGURE 10
m-D-vertices, M-D-vertices and flattenings viewed in the curve $\widetilde{\mathbf{D}}$

The points of $\widetilde{\mathbf{D}}$ at which $\frac{d\vartheta}{ds} = 0$ are the Darboux vertices of γ. The points of $\widetilde{\mathbf{D}}$ at which $\vartheta = 0$ are the flattenings of γ. The M-D-vertices of γ are the critical points of $\vartheta(s)$ whose convex side is pointing towards the axis $\vartheta = 0$. The m-D-vertices of γ are the critical points of $\vartheta(s)$ whose convex side is pointing against the axis $\vartheta = 0$. The flattenings of γ are the crossings of $\widetilde{\mathbf{D}}$ with the axis $\vartheta = 0$, see Figure 10.

The biflattening perestroika of a generic family of curves is represented in Figure 11.

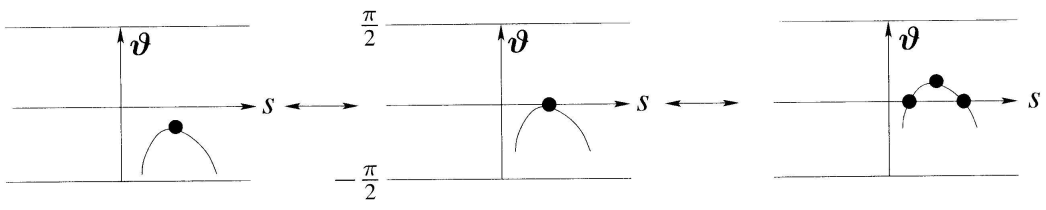

FIGURE 11

The biflattening perestroika of a generic family of curves

The inflection perestroika of a generic family of curves is represented in Figure 12.

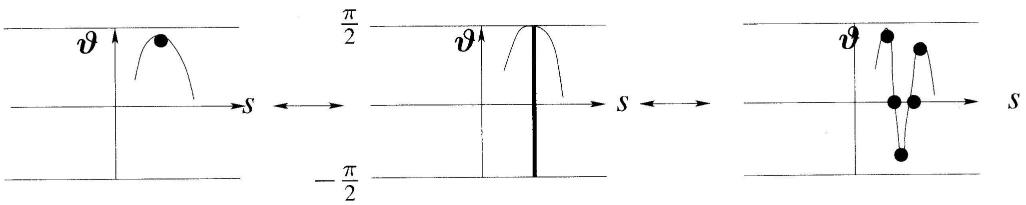

FIGURE 12

At an inflection perestroika of a generic family of curves a vertical double segment appears

Although the Frenet trihedral is not uniquely defined at an inflection point (see Main Remark, §2), the rectifying plane has a limit position at that point. This limit plane is parallel to the great circle which is 'added' to the Darboux indicatrix during an inflection perestroika (see Figure 4). The presence of this circle means that every direction in this limit plane defines a Darboux vector at the inflection point. Thus at the inflection moment a vertical double segment — corresponding to all angles in $[-\pi/2, \pi/2]$, counted twice — is added to the curve $\widetilde{\mathbf{D}}$ in the angle-length strip. After the moment of perestroika this double segment splits and intersects the axis $\vartheta = 0$ in two points (corresponding to two new flattenings).

The evolution of the curve $\widetilde{\mathbf{D}}$ during a double Darboux vertex perestroika in a generic family of curves is represented in Figure 13. The flattenings are not involved in the double Darboux vertex perestroika.

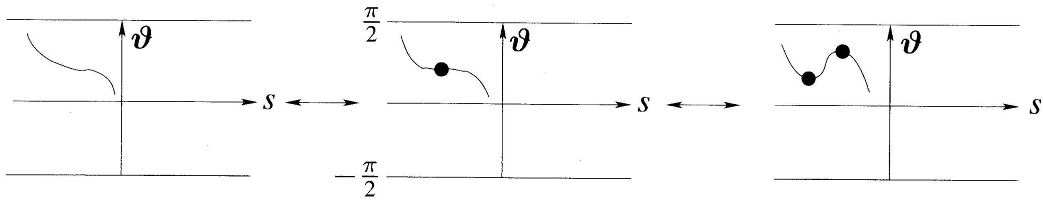

FIGURE 13
The double Darboux vertex perestroika of a generic family of curves

The other possibility of a biflattening (inflection or double Darboux vertex) perestroika can be obtained from Figure 11 (12 or 13, respectively) by mirroring with respect to the horizontal axis.

8. DARBOUX VERTICES AND GEOMETRY OF THE FOCAL CURVE

An *osculating sphere* at a point of a curve in Euclidean space $\mathbf{R}^3$ is a sphere having at least 4-point contact with the curve at that point (the affine subspaces are considered as spheres of infinite radius). *The osculating sphere is unique at any point of a generic curve.*

A *vertex* of a curve in $\mathbf{R}^3$ is a point where the curve has at least 5-point contact with an osculating sphere at that point.

Let $\gamma\colon s \mapsto \gamma(s)$ be a generic curve in Euclidean space $\mathbf{R}^3$. Write $C_\gamma(s)$ for the centre of the osculating sphere of γ at s. If s is a flattening of γ then $C_\gamma(s)$ is not defined and we will say that $C_\gamma(s)$ is at infinity. A non generic curve can have an inflection; in this case the centre of the osculating sphere is not defined and we say that it is at infinity.

DEFINITION. The curve $C_\gamma\colon s \mapsto C_\gamma(s)$ consisting of the centres of the osculating spheres of γ is called the *focal curve of* γ (for a more general study see [17]).

THEOREM 5. *A curve γ of $\mathbf{R}^3$ and its focal curve C_γ have the same Darboux indicatrix:* $(\mathbf{D}(C_\gamma))(s) = (\mathbf{D}(\gamma))(s)$. *This statement is also valid for the points where the curve γ (or C_γ) has an ordinary cusp. (The definition of Darboux vector at these points is given in the proof.)*

We reformulate the definition of an inflection of a curve in $\mathbf{R}^3$ in terms of the tangent indicatrix:

DEFINITION 8.1. An *inflection* of a curve immersed in $\mathbf{R}^3$ is a point where the first derivative of its tangent indicatrix vanishes (i.e. it is a cusp of the tangent indicatrix for a generic inflection).

Although the focal curve C_γ is not defined when the curve γ has a flattening, the tangent indicatrix of C_γ is always well defined. We will say that C_γ has an inflection or a flattening if its tangent indicatrix has a cusp or a spherical inflection, respectively.

THEOREM 6. (a) *The Darboux vertices of γ correspond to the Darboux vertices of C_γ.*

(b) *The vertices of γ correspond to cusps of C_γ.*

(c) *The flattenings of γ correspond to the inflections of C_γ. The inflections of C_γ are at infinity.*

(d) *The inflections of γ correspond to the flattenings of C_γ. The flattenings of C_γ are at infinity.*

So the finite part of the focal curve has neither inflections nor flattenings.

REMARK. Item (d) holds only under Fenchel's convention, (b), in §2.1.

COROLLARY 1. *The focal curve of a generic closed curve in Euclidean space $\mathbf{R}^3$ has an even number of cusps, has no flattening and has (as a stable property) an even number of isolated inflections (at infinity!).*

The following theorem is a by-product of the proof of Theorem 5:

FOCAL CURVE THEOREM. *Let $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ be a smoothly immersed curve without vertices and with curvature κ and torsion τ both nowhere zero. Then the focal curve C_γ has curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ both nowhere zero and*

$$\left|\frac{\tau}{\kappa}\cdot\frac{\tilde{\tau}}{\tilde{\kappa}}\right| = 1.$$

A *helix* of $\mathbf{R}^3$ is a curve for which the function τ/κ is constant. So, in [10], [13] and [12] it is stated that *at the points of local extremum of the function τ/κ the curve behaves like a helix.*

COROLLARY 2. *The focal curve of a helix with curvature and torsion nowhere zero is also a helix with curvature and torsion nowhere zero.*

8.1 PROOF OF THEOREMS 5 AND 6

Let γ be a curve in Euclidean space $\mathbf{R}^3$, parametrised by arc length s. Let R be the radius of curvature of γ, i.e. $R = 1/\kappa$. Write C_γ for the focal curve of γ. Derivation with respect to s is denoted by a prime: $d/ds = {}'$.

LEMMA. *Write $\beta = R'/\tau$. The derivative of the focal curve with respect to s is $C_\gamma'(s) = ((R\tau + \beta')\mathbf{b})(s)$.*

Proof. A direct and easy calculation shows that $C_\gamma(s) = (\gamma + R\mathbf{n} + \beta\mathbf{b})(s)$. So, derivation with respect to s gives:

$$C_\gamma'(s) = (\mathbf{t} + R(-\kappa\mathbf{t} + \tau\mathbf{b}) + R'\mathbf{n} + \beta(-\tau\mathbf{n}) + \beta'\mathbf{b})(s) = ((R\tau + \beta')\mathbf{b})(s). \qquad \square$$

Proof of Theorem 5. At an ordinary cusp of a curve, the tangent line is well defined and we define the unit tangent vector to the curve in a neighbourhood of the cusp as a continuous vector field of unit tangent vectors on thc curvc. Of course, the orientation of the curve is different from the orientation of the unit tangent vector along one of the branches of the curve. The tangent plane is also well defined at an ordinary cusp and the binormal unit vector is defined as the unit vector normal to this tangent plane, which is locally on the same side of the plane as the curve. The unit tangent vector and the binormal unit vector define automatically the unit normal vector. So in the neighbourhood of the cusp we have a unit trihedral which differs from the Frenet trihedral, on one of the branches of the curve, only in the sign of some of the unit vectors. So at the points in which the Frenet trihedral is well defined, the Darboux axis coincides with the instantaneous axis of rotation of the unit trihedral just defined. By the preceding lemma we can say that $\mathbf{b}$ is the unit tangent vector $\tilde{\mathbf{t}}$ of the focal curve C_γ. By the Frenet equations we have $\mathbf{b}' = -\tau\mathbf{n}$. So the unit normal and binormal vectors of C_γ are, up to the sign, $\mathbf{n}$ and $\mathbf{t}$, respectively. This implies that γ and C_γ have the same Darboux axis. $\square$

Proof of Theorem 6 a). Direct corollary of Proposition 1. $\square$

Proof of Theorem 6 b). The preceding lemma implies that $C_\gamma'(s) = 0$ if and only if $(R\tau + \beta')(s) = 0$. The points of γ for which $(R\tau + \beta')(s) = 0$ are its vertices (see [7]). In fact a spherical curve identically satisfies the equation $(R\tau + \beta')(s) = 0$. Thus the vertices of γ correspond to the cusps of C_γ. $\square$

Proof of Theorem 6 c). The statement follows from Definition 8.1 applied to C_γ, Lemma 2.4 and Fenchel's statement (2). □

Proof of Theorem 6 d). The statement follows from Definition 8.1 applied to γ, Fenchel's statement (2) and Lemma 2.4. □

PART II. CURVES IN EUCLIDEAN n-SPACE

9. Darboux vertices and twistings

PROPOSITION D-F. *The Darboux vertices of a smooth immersed curve in* $\mathbf{R}^3$ *coincide with the points at which its tangent indicatrix has a flattening.*

Proof. First, the tangent indicatrix of $\mathbf{T}$ is $\mathbf{N}$. By Lemma 2.4, a flattening of $\mathbf{T}$ corresponds to a spherical inflection of $\mathbf{N}$. Next, by Fenchel's statement (4), the Darboux vertices of γ correspond to the spherical inflections of $\mathbf{N}$. Thus Darboux vertices of γ and flattenings of $\mathbf{T}$ coincide. □

From Proposition D-F we see that there are at least two direct generalisations of Darboux vertices for higher dimensional spaces:

DEFINITION OF A DARBOUX VERTEX. When the Frenet frame of a curve in Euclidean space is translated to the origin, it determines a rigid motion. If the space is of odd dimension then this rigid motion has an instantaneous axis of rotation, called the *Darboux axis*. A point of a curve for which its Darboux axis is stationary is called a *Darboux vertex* of that curve.

DEFINITION OF A TWISTING. The *tangent indicatrix* of a curve in Euclidean space $\mathbf{R}^n$, $n > 2$, is the curve $\mathbf{T}$ on $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ described by the unit tangent vector of that curve, translated to the origin. A point of a curve γ in $\mathbf{R}^n$, $n > 2$, is called a *twisting* ([13]) if the tangent indicatrix of γ considered as a spatial curve has a flattening at the corresponding point.

PROPOSITION D-T. *For every smooth curve Darboux vertices and twistings coincide only in Euclidean* 3*-space.*

Proof. Let γ be a curve in $\mathbf{R}^n$. A vector belongs to the Darboux axis of γ (when it exists, i.e. n odd) if and only if it is orthogonal to the hyperplane spanned by the derivatives of the unit vectors of the Frenet

n-hedral: $\mathbf{t}'$, $\mathbf{n}_1'$, ..., $\mathbf{n}_{n-1}'$. A vector is orthogonal to the osculating hyperplane of the tangent indicatrix of γ at a point if and only if it is orthogonal to the hyperplane spanned by the derivatives of $\mathbf{t}$ at that point: $\mathbf{t}'$,..., $\mathbf{t}^{(n-1)}$. These hyperplanes coincide for every value of the curve's parameter only for $n = 3$. That is, the Darboux axis of γ is orthogonal to the osculating hyparplane of the tangent indicatrix of γ (for every value of the curve's parameter) only if $n = 3$. Thus, for any curve γ the stationariness of its Darboux axis implies and is implied by the stationariness of the osculating hyperplane of the tangent indricatrix only if $n = 3$.

For odd $n > 3$, the Darboux axis of a generic curve γ can be orthogonal to the osculating hyperplane of the tangent indicatrix only at isolated points: the flattenings of γ (see first remark of § 10 or the proof of Proposition 3 in § 11.1). At these points, the Darboux axis of γ and the osculating hyperplane of the tangent indicatrix are both non stationary. Of course, the coincidence of Darboux vertices and twistings for odd $n > 3$ is possible for very degenerate (i.e. non generic) curves.

Necessary and sufficient conditions for the stationariness of the Darboux axis of a curve in $\mathbf{R}^{2k+1}$, $k \geq 1$, are given in Theorem 7 below. □

In § 10 and § 11 we present theorems on Darboux vertices and twistings, respectively.

10. Darboux vertices of curves in Euclidean space $\mathbf{R}^n$

THEOREM 7. *Let γ be a smoothly immersed curve in $\mathbf{R}^{2k+1}$, $k \geq 1$. Write $\kappa_1, \kappa_2, \ldots, \kappa_{2k}$ for its curvatures. The curve γ has a Darboux vertex at $s = s_0$ if and only if $(\kappa_2/\kappa_1)' = 0$, $(\kappa_4/\kappa_3)' = 0$, ..., $(\kappa_{2k}/\kappa_{2k-1})' = 0$ at $s = s_0$.*

COROLLARY. *A generic curve in $\mathbf{R}^{2k+1}$, $k \geq 2$, has no Darboux vertex.*

Suppose that $n = 2k + 1$. Write $\mathbf{t}(s), \mathbf{n}_1(s), \ldots, \mathbf{n}_{2k}(s)$ for the unit vectors of the Frenet n-hedral of γ at s.

PROPOSITION 1. *The Darboux axis of γ at time s is determined by the kernel of the Frenet matrix $M(s)$, given with respect to the basis $\mathbf{t}$, $\mathbf{n}_1$, ..., $\mathbf{n}_{2k}$ (see $M(s)$ in equation $(*)$ below). See also Lemma 2.1 in subsection 2.3.*

Proof. This is a direct consequence of the Frenet equations. □

The curvatures $\kappa_1, \ldots, \kappa_{2k-1}$ of a generic curve in $\mathbf{R}^{2k+1}$ are positive everywhere. The last curvature κ_{2k} can vanish at isolated points (the flattenings). Write

$$a_0 = \kappa_2\kappa_4 \cdots \kappa_{2k}\,, \quad a_1 = \frac{\kappa_1}{\kappa_2}a_0\,, \quad a_2 = \frac{\kappa_3}{\kappa_4}a_1\,, \ldots,$$
$$a_j = \frac{\kappa_{2j-1}}{\kappa_{2j}}a_{j-1}\,, \ldots, \quad a_k = \frac{\kappa_{2k-1}}{\kappa_{2k}}a_{k-1} = \kappa_1\kappa_3\cdots\kappa_{2k-1}\,.$$

We define the *Darboux vector* of a curve in $\mathbf{R}^{2k+1}$ by

$$\tilde{\mathbf{d}} = a_0\mathbf{t} + a_1\mathbf{n}_2 + \cdots + a_k\mathbf{n}_{2k}\,.$$

PROPOSITION 2. *The Darboux vector $\tilde{\mathbf{d}} = a_0\mathbf{t} + a_1\mathbf{n}_2 + \cdots + a_k\mathbf{n}_{2k}$ lies in the kernel of $M(s)$ and, if the curve is generic, it generates the kernel of $M(s)$.*

Proof. Direct verification shows that the following equality holds:

$$(*) \qquad M(s)\cdot\tilde{\mathbf{d}} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 \\ 0 & -\kappa_2 & 0 & & & \vdots \\ 0 & 0 & & & & \\ \vdots & & & & 0 & \kappa_{2k} \\ 0 & 0 & & \cdots & -\kappa_{2k} & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ 0 \\ a_1 \\ 0 \\ \vdots \\ 0 \\ a_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In fact the inner product of $\tilde{\mathbf{d}}$ with the odd row vectors of M yields zero since the odd row vectors of M are linear combinations of the unit vectors $\mathbf{n}_j$ for odd j. Denote $\mathbf{t} = \mathbf{n}_0$. The $2j^{\text{th}}$ row vector of M has the form

$$\kappa_{2j+1}\mathbf{n}_{2j} - \kappa_{2j+2}\mathbf{n}_{2j+2}\,, \quad j = 0, \ldots, k-1\,,$$

so its inner product with $\tilde{\mathbf{d}}$ is

$$a_j\kappa_{2j+1} - a_{j+1}\kappa_{2j+2} = a_j\kappa_{2j+1} - \frac{\kappa_{2j+1}}{\kappa_{2j+2}}a_j\kappa_{2j+2} = 0\,. \qquad \square$$

REMARK. At the flattenings of γ, the Darboux axis contains the $2k$-normal vector $\mathbf{n}_{2k}$. This follows from the definition of the functions $a_0, \ldots, a_k$.

10.1 Proof of Theorem 7

The Darboux vector $\tilde{\mathbf{d}} = a_0\mathbf{t}+a_1\mathbf{n}_2+\cdots+a_k\mathbf{n}_{2k}$ generates the Darboux axis of γ. The derivative of the vector $\tilde{\mathbf{d}}$, with respect to arc length s, $\tilde{\mathbf{d}}' = d\tilde{\mathbf{d}}/ds$, can be decomposed into two components, one of them orthogonal to $\tilde{\mathbf{d}}$ and the other parallel to $\tilde{\mathbf{d}}$. The line generated by $\tilde{\mathbf{d}}$ is stationary at $s = s_0$ if and only if the component of $\tilde{\mathbf{d}}'(s_0)$ orthogonal to $\tilde{\mathbf{d}}(s_0)$ is zero, i.e. if and only if $\tilde{\mathbf{d}}'(s_0)$ is proportional to $\tilde{\mathbf{d}}(s_0)$. So we must first calculate $\tilde{\mathbf{d}}'$.

Lemma 10.1. $\tilde{\mathbf{d}}' = a_0'\mathbf{t} + a_1'\mathbf{n}_2 + \cdots + a_k'\mathbf{n}_{2k}$.

Proof. Evidently, $\tilde{\mathbf{d}}' = (a_0'\mathbf{t}+a_1'\mathbf{n}_2+\cdots+a_k'\mathbf{n}_{2k})+(a_0\mathbf{t}'+a_1\mathbf{n}_2'+\cdots+a_k\mathbf{n}_{2k}')$. Write X for the second term on the right hand side of this equation. First, to prove that $X \equiv 0$, we will use the Frenet equations:

$$X = a_0(\kappa_1\mathbf{n}_1) + a_1(-\kappa_2\mathbf{n}_1 + \kappa_3\mathbf{n}_3) + \cdots \\ + a_j(-\kappa_{2j}\mathbf{n}_{2j-1} + \kappa_{2j+1}\mathbf{n}_{2j+1}) + \cdots + a_k(-\kappa_{2k}\mathbf{n}_{2k-1}).$$

Next, for $j = 1,\ldots,k$, we will use the definition of the functions a_j:

$$\begin{aligned} a_j(-\kappa_{2j}\mathbf{n}_{2j-1} + \kappa_{2j+1}\mathbf{n}_{2j+1}) &= -(\kappa_{2j-1}/\kappa_{2j})a_{j-1}\kappa_{2j}\mathbf{n}_{2j-1} + a_j\kappa_{2j+1}\mathbf{n}_{2j+1} \\ &= -a_{j-1}\kappa_{2j-1}\mathbf{n}_{2j-1} + a_j\kappa_{2j+1}\mathbf{n}_{2j+1}. \end{aligned}$$

Using these equalities, the sum of two neighbouring terms of X,

$$a_j(-\kappa_{2j}\mathbf{n}_{2j-1} + \kappa_{2j+1}\mathbf{n}_{2j+1}) + a_{j+1}(-\kappa_{2(j+1)}\mathbf{n}_{2(j+1)-1} + \kappa_{2(j+1)+1}\mathbf{n}_{2(j+1)+1}),$$

becomes

$$(-a_{j-1}\kappa_{2j-1}\mathbf{n}_{2j-1} + a_j\kappa_{2j+1}\mathbf{n}_{2j+1}) + (-a_j\kappa_{2j+1}\mathbf{n}_{2j+1} + a_{j+1}\kappa_{2(j+1)+1}\mathbf{n}_{2(j+1)+1}).$$

This implies that $X \equiv 0$. Thus $\tilde{\mathbf{d}}' = a_0'\mathbf{t} + a_1'\mathbf{n}_2 + \cdots + a_k'\mathbf{n}_{2k}$. □

Proof of Theorem 7. By Lemma 10.1, $\tilde{\mathbf{d}}'$ is parallel to $\tilde{\mathbf{d}}$ at a point if and only if

$$\frac{a_0'}{a_0} = \frac{a_1'}{a_1} = \cdots = \frac{a_k'}{a_k}$$

at that point. These equalities hold simultaneously if and only if the following equalities hold simultaneously:

$$\frac{\kappa_1'}{\kappa_1} = \frac{\kappa_2'}{\kappa_2}, \quad \frac{\kappa_3'}{\kappa_3} = \frac{\kappa_4'}{\kappa_4}, \quad \ldots, \quad \frac{\kappa_{2k-1}'}{\kappa_{2k-1}} = \frac{\kappa_{2k}'}{\kappa_{2k}},$$

that is, if and only if

$$(\kappa_2/\kappa_1)' = 0, \quad (\kappa_4/\kappa_3)' = 0, \quad \ldots, \quad (\kappa_{2k}/\kappa_{2k-1})' = 0. \quad □$$

REMARK. For generic curves in Euclidean space $\mathbf{R}^{2k}$ the Frenet $2k$-hedral has no axis of rotation and the Frenet matrix degenerates only at flattenings. The degeneracy or non-degeneracy of the Frenet matrix depends only on the odd curvatures: *the determinant of the Frent matrix is equal to* $\kappa_1^2\kappa_3^2\cdots\kappa_{2k-1}^2$.

This means that even when all even curvatures vanish the Frenet matrix will be non-degenerate if the odd curvatures don't vanish!

11. TWISTINGS OF CLOSED CURVES IN EUCLIDEAN n-SPACE

A curve $\alpha\colon \mathbf{R} \to \mathbf{R}^n$, $n > 2$, whose tangent vector forms a constant angle with a given direction v in $\mathbf{R}^n$ is called a *helix.* Thus *all points of a helix are twistings.*

Any generic closed curve immersed in $\mathbf{R}^3$ *has at least two twistings.*

At a twisting of a curve, there exist helices having at least $(n+2)$*-point contact with it, whereas at an ordinary point of a generic curve a helix can have at most* $(n+1)$*-point contact with it (see [13]).*

The first statement follows from the fact that a smoothly immersed closed curve in the two-dimensional sphere has at least 2 flattenings.

PROBLEM 1 (C. Romero-Fuster, 1999). Does the fact that any generic smoothly immersed closed curve in $\mathbf{R}^3$ has at least two twistings generalise to closed curves in $\mathbf{R}^n$?

PROBLEM 2. Find conditions on a closed curve in $\mathbf{R}^n$ that guarantee that the number of twistings is not less than some lower bound depending on n.

To answer these problems we present the following global theorems:

THEOREM 8. *The number of twistings of a closed curve in* $\mathbf{R}^{2k+1}$ *is at least equal to the number of its flattenings.*

THEOREM 9. *There exist closed curves in* $\mathbf{R}^{2k}$, $k \geq 2$, *without twistings.*

DEFINITION. A closed curve in $\mathbf{R}^n$ is called a *Barner curve* if for every $(n-1)$-tuple of (not necessarily geometrically different) points of the curve there exists a hyperplane through these points that does not intersect the curve elsewhere.

THEOREM 10. *Any Barner curve in* $\mathbf{R}^{2k+1}$ *has at least* $2k+2$ *twistings.*

Proof. Theorem 10 follows from Theorem 8 and from the fact ([6]) that any Barner curve in $\mathbf{R}^{2k+1}$ has at least $2k+2$ flattenings.

A hypersphere of $\mathbf{S}^n$ of maximal radius will be called an *equator*. In relation with Problem 2, it was proved in [13] that:

A generic closed curve in $\mathbf{R}^{2k+1}$ *whose tangent indicatrix meets each equator of* $\mathbf{S}^{2k}$ *in at most* $2k$ *points (counting their multiplicities) has at least* $2k+2$ *twistings.*

REMARK. This statement is true, but unfortunately it is empty: *there is no closed curve satisfying the required conditions.*

Proof of the remark. First, any closed curve of $\mathbf{S}^{2k}$ which meet each equator in at most $2k$ points (counting their multiplicities) must lie in an open hemisphere of $\mathbf{S}^{2k}$, [15]. Next, the tangent indicatrix of a closed curve in $\mathbf{R}^{2k+1}$ is a closed curve on the sphere $\mathbf{S}^{2k}$ intersecting each equator at least twice [10]:

1° Given an equator E of $\mathbf{S}^{2k}$, there is a unique hyperplane $H_E \subset \mathbf{R}^{2k+1}$ containing it: $E = H_E \cap \mathbf{S}^{2k}$.

2° The orthogonal projection of γ on the line orthogonal to H_E has at least two critical points.

3° The tangent vector of γ at each one of these critical points is contained in a hyperplane parallel to H_E.

Thus at these points the tangent indicatrix of γ intersects the equator E. □

Proof of Theorem 9. We indicate some closed curves without twistings:

PROPOSITION. *The closed curve* $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^{2k}$*, given by*

$$\gamma(\vartheta) = (\cos\vartheta, \sin\vartheta, \cos 2\vartheta, \sin 2\vartheta, \dots, \cos k\vartheta, \sin k\vartheta)\,,$$

has no twisting.

Proof. The tangent indicatrix of γ is the spherical curve given by

$$\mathbf{T}(\vartheta) = \frac{1}{\sqrt{M}}(-\sin\vartheta, \cos\vartheta, -2\sin 2\vartheta, 2\cos 2\vartheta, \ldots, -k\sin k\vartheta, k\cos k\vartheta),$$

where $M = 1+4+\cdots+k^2$. The Wronski determinant of the curve $\mathbf{T}$ never vanishes (in fact it is constant). So $\mathbf{T}$ has no flattening and thus γ has no twisting.

Another proof. All curvatures of $\mathbf{T}$ are constant and $\mathbf{T}$ does not lie in a hyperplane. $\square$

REMARK. Any small enough perturbation of γ (taking the derivatives into account) has no twisting.

11.1 PROOF OF THEOREM 8

Throughout this section, a *closed curve* in $\mathbf{R}^n$ always means a smooth immersion $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^n$. An immersion is *good* if the derivatives of γ of orders $(1, \ldots, n-1)$ are linearly independent at any point. *A generic immersion is good.*

Theorem 8 follows from the following proposition:

PROPOSITION 3. *Between two consecutive flattenings of a good curve in the Euclidean space* $\mathbf{R}^{2k+1}$ *there is at least one twisting.*

To prove Proposition 3 we will need Proposition 4 below. We recall some definitions related to the geometry of spherical curves:

An *osculating* $(n-1)$*-sphere* at a point of a spherical curve $\gamma \subset \mathbf{S}^n \subset \mathbf{R}^{n+1}$ is an $(n-1)$-dimensional sphere having at least $(n+1)$-point contact with the curve at that point. *Each point of a generic spherical curve has a unique osculating hypersphere.*

A *vertex* of a spherical curve in $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ is a point at which the curve has at least $(n+2)$-point contact with an osculating hypersphere at that point. If the curve has $(n+2l)$-point contact with its osculating hypersphere, $l \geq 1$, the point is a *vertex of odd multiplicity* $2l-1$.

An *osculating equator* at a point of a spherical curve in $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ is an equator having at least n-point contact with the curve at that point. *Each point of a generic spherical curve has a unique osculating equator.*

A *spherical inflection* of a spherical curve in $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ is a point at which the curve has at least $(n+1)$-point contact with an osculating equator at that point. If the curve has exactly $(n+1)$-point contact with its osculating equator then the point is called a *simple spherical inflection.*

PROPOSITION 4. *Between two consecutive spherical inflections of a spherical curve $\gamma \subset \mathbf{S}^{2k} \subset \mathbf{R}^{2k+1}$ there is an odd number of vertices of odd multiplicity (hence at least one).*

LEMMA [15]. *The vertices of a spherical curve $\gamma \subset \mathbf{S}^n \subset \mathbf{R}^{n+1}$ are the flattenings of γ regarded as a spatial curve.*

Proof of Proposition 3. Let $\mathbf{T}$ be the tangent indicatrix of $\gamma \subset \mathbf{R}^{2k+1}$. The hyperplane through the origin containing the 2-codimensional osculating space of $\mathbf{T}$ is parallel to the osculating hyperplane of γ: it contains $\mathbf{T}$ and the subspace of codimension 2 generated by the derivatives of $\mathbf{T}$. The intersection of this hyperplane with $\mathbf{S}^{2k}$ is the osculating equator of $\mathbf{T}$. If the osculating hyperplane of γ is stationary then the osculating equator of $\mathbf{T}$ is stationary. So the flattenings of γ correspond to the spherical inflections of its tangent indicatrix $\mathbf{T}$. Proposition 4 and the preceding lemma imply that between two consecutive flattenings of γ there is at least one twisting. $\square$

Thus, it remains only to prove Proposition 4.

LEMMA 1. *At a spherical inflection of a spherical curve $\gamma \subset \mathbf{S}^n \subset \mathbf{R}^{n+1}$ the osculating hypersphere coincides with the osculating equator.*

Proof. At a spherical inflection the osculating equator is a hypersphere having at least $(n+1)$-point contact with the curve, so it is osculating. $\square$

LEMMA 2. *At the simple spherical inflections of a spherical curve $\gamma \subset \mathbf{S}^{2k} \subset \mathbf{R}^{2k+1}$ the curve passes from one side of the osculating equator to the other.*

Proof. At the simple spherical inflections the order of contact of the curve with its osculating equator is odd. $\square$

LEMMA 3. *If a spherical curve $\gamma \subset \mathbf{S}^{2k}$ has even order of contact with its osculating hypersphere at a given point then this point is a vertex of odd multiplicity of the curve.*

Proof. A spherical curve $\gamma \subset \mathbf{S}^{2k}$ has $(2k+1)$-point contact with its osculating hypersphere at a non-vertex. Thus if the order of contact is $2k+2l$ the point is a vertex of odd multiplicity. □

11.1.1 PROOF OF PROPOSITION 4

Let $\gamma\colon \mathbf{R} \to \mathbf{S}^{2k} \subset \mathbf{R}^{2k+1}$ be a spherical curve in the $2k$-dimensional sphere of radius R. Suppose that $\gamma(t_0)$ and $\gamma(t_1)$ are two consecutive simple spherical inflections of γ, with $t_0 < t_1$, and there is no spherical inflection in the open interval (t_0, t_1). By Lemma 3, we need only to show that there is a point $\tilde{t} \in (t_0, t_1)$, such that the curve γ has even order of contact with its osculating hypersphere at $\gamma(\tilde{t})$.

For each $t \in (t_0, t_1)$ the radius of the osculating hypersphere of γ at $\gamma(t)$ is smaller than R. So each osculating hypersphere separates $\mathbf{S}^{2k}$ into two discs of different size. We distinguish these two discs: the smaller one will be called Int and the larger one will be called Ext. This permits one to co-orient continuously the osculating hyperspheres of γ for $t \in (t_0, t_1)$: we choose as co-orienting vector the unit vector tangent to $\mathbf{S}^{2k}$, orthogonal to the osculating hypersphere and pointing from Ext(t) to Int(t). By continuity we extend this co-orientation to the osculating hyperspheres of γ at the spherical inflections $\gamma(t_0)$ and $\gamma(t_1)$.

At each point $\gamma(t)$, with $t \in (t_0, t_1)$, the osculating equator separates $\mathbf{S}^{2k}$ into two hemispheres. The hemisphere containing the disc Int(t) will be denoted by INT(t) and the other will be denoted by EXT(t). This permits one to co-orient continuously the osculating equators of γ, for $t \in (t_0, t_1)$: we choose as co-orienting vector the unit vector tangent to $\mathbf{S}^{2k}$, orthogonal to the osculating equator and pointing from EXT(t) to INT(t). By continuity we extend this co-orientation to the osculating equators of γ at the spherical inflections $\gamma(t_0)$ and $\gamma(t_1)$.

LEMMA 4. *The co-orienting vectors of the osculating hyperspheres and of the osculating equators of the spherical curve γ coincide for $t \in [t_0, t_1]$.*

Proof. They coincide by construction. □

LEMMA 5. *At* $\gamma(t_0)$ *the curve* γ *traverses its osculating equator from* $\mathrm{EXT}(t_0)$ *to* $\mathrm{INT}(t_0)$. *At* $\gamma(t_1)$ *the curve* γ *traverses its osculating equator from* $\mathrm{INT}(t_1)$ *to* $\mathrm{EXT}(t_1)$.

Proof. By Lemma 2, the spherical curve γ traverses its osculating equator at the simple spherical inflections $\gamma(t_0)$ and $\gamma(t_1)$.

If a point of a smooth spherical curve $\gamma \subset \mathbf{S}^{2k}$ is not a spherical inflection then the order of contact of the curve with its osculating equator is even and equal to $2k$. Thus at each point $\gamma(t)$, with $t \in (t_0, t_1)$, the order of contact of γ with its osculating equator is even. So at each point $\gamma(t)$, with $t \in (t_0, t_1)$, the curve lies locally in one of the two hemispheres determined by the osculating equator.

The definition of $\mathrm{INT}(t)$ implies that for each $t \in (t_0, t_1)$ the osculating hypersphere of γ at $\gamma(t)$ is contained in the hemisphere $\mathrm{INT}(t)$. At each point $\gamma(t)$, with $t \in (t_0, t_1)$, the curve must lie locally in the same hemisphere as its osculating hypersphere. Thus at each point $\gamma(t)$, with $t \in (t_0, t_1)$, the curve γ must lie locally in $\mathrm{INT}(t)$. This implies that at $\gamma(t_0)$ the curve γ traverses its osculating equator from $\mathrm{EXT}(t_0)$ to $\mathrm{INT}(t_0)$ and that at $\gamma(t_1)$ the curve γ traverses its osculating equator from $\mathrm{INT}(t_1)$ to $\mathrm{EXT}(t_1)$. □

LEMMA 6. *At* $\gamma(t_0)$ *the curve* γ *traverses its osculating hypersphere from* $\mathrm{Ext}(t_0)$ *to* $\mathrm{Int}(t_0)$. *At* $\gamma(t_1)$ *the curve* γ *traverses its osculating hypersphere from* $\mathrm{Int}(t_1)$ *to* $\mathrm{Ext}(t_1)$.

Proof. This is a corollary of Lemma 1, Lemma 4 and Lemma 5. □

By Lemma 6, at $t = t_0 + \varepsilon$ (and at $t = t_1 - \varepsilon$) , with $\varepsilon > 0$ small enough, the curve traverses the osculating hypersphere from $\mathrm{Ext}(t)$ to $\mathrm{Int}(t)$ (from $\mathrm{Int}(t)$ to $\mathrm{Ext}(t)$, respectively). This implies that there exists an odd number of points and at least one point $\gamma(\tilde{t})$, with $\tilde{t} \in (t_0, t_1)$, at which the curve γ does not traverse its osculating hypersphere. Thus the order of contact of the curve with its osculating hypersphere at $\gamma(\tilde{t})$ is even or infinite. Thus, by Lemma 3, the point $\gamma(\tilde{t})$ is a vertex of odd order of the spherical curve γ. Proposition 4 is proved for the case of two consecutive simple spherical inflections.

If, accidentally, the spherical inflections $\gamma(t_0)$ and $\gamma(t_1)$ are not simple then we can make a small enough perturbation Γ (taking derivatives into account) of the curve γ in order to decompose the spherical inflections into a finite number of simple ones. The simple spherical inflections of Γ will be grouped inside two small neighbourhoods U_0 and U_1 of $\gamma(t_0)$ and $\gamma(t_0)$, respectively. Denote by $\hat{t}_0$ the largest value of t for which the curve Γ has a simple spherical inflection in U_0. Denote by $\hat{t}_1$ the smallest value of t for which the curve Γ has a simple spherical inflection in U_1. We proved that between $\Gamma(\hat{t}_0)$ and $\Gamma(\hat{t}_1)$ there is at least one vertex of Γ of odd order. We can join Γ to γ by a homotopy γ_s such that at each $s \in [0,1)$ the curve γ_s has only simple spherical inflections. Thus each γ_s will have at least one vertex of odd order. This implies that γ will also have at least one vertex of odd order. This proves Proposition 4. $\square$

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