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4. Tile homotopy groups

We establish some notation. Let P be the free group on generators x and y. We may think of elements of P as paths on the square lattice. For a protoset $\mathcal{T}$, let $Z(\mathcal{T})$ be the smallest normal subgroup of P that contains all boundary words of tiles in $\mathcal{T}$. Corollary 3.6 strongly suggests the following definition.

DEFINITION 4.1. The *tile path group* of $\mathcal{T}$, denoted $P(\mathcal{T})$, is the quotient group $P/Z(\mathcal{T})$.

Corollary 3.6 says that if a region R has a tiling by $\mathcal{T}$, then its boundary word is trivial in the tile path group. However, it may be difficult to work with the tile path group directly. Theorem 3.9 above uses a representation of the tile path group to show that the boundary word of a $(3m+1)\times(3n+1)$ rectangle is non-trivial.

NOTATION 4.2. From now on, we will use x and y to denote the free generators of P, and $\bar{x}, \bar{y}$ to denote their images in $P(\mathcal{T})$.

The tile path group turns out to be "too big" in a sense. Let $C \subseteq P$ be the subgroup of *closed words*, i.e. those words that correspond to a closed path. (For the square lattice, this is simply the commutator subgroup of P; for other lattices it won't necessarily be so.) It is clear that C is indeed a subgroup, and is normal in P. Also, since every boundary word is a closed word, we have $Z(\mathcal{T}) \subseteq C$. Now we have the insightful definition of Conway and Lagarias.

DEFINITION 4.3. The *tile homotopy group* of $\mathcal{T}$ is the quotient $\pi(\mathcal{T}) = C/Z(\mathcal{T})$.

The relevance of this group is two-fold. Firstly, we are interested in the boundary word of a region, modulo $Z(\mathcal{T})$. Since every boundary word is closed, only elements of $\pi(\mathcal{T})$ need to be considered. Secondly, there is a strong connection between the tile homotopy group and the tile homology group, which we now examine.

RELATION BETWEEN TILE HOMOTOPY AND TILE HOMOLOGY

To understand the tile homotopy group, we first seek an understanding of the group C of closed words. This is a subgroup of the free group P, so the following classical result of Nielsen and Schreier is relevant. See [9, Chapter 7, Section 2] for more about this. We sketch its proof, because we are interested in producing an explicit set of free generators of C.

THEOREM 4.4. *Any subgroup of a free group is also free.*

Proof (sketch). Let G be a free group and let $H \subseteq G$ be a subgroup. Then G is the fundamental group of a bouquet of circles, X, one circle for each generator. Subgroups of G are in one-to-one correspondence with (connected) covering spaces of X. Thus H corresponds to a covering space $Y \to X$, where elements of H are exactly those closed paths on X that lift to closed paths on Y, and the fundamental group of Y is precisely H. Since Y is a graph (i.e. a 1-dimensional CW complex), the proof is finished by the following proposition.

PROPOSITION 4.5. *The fundamental group of a graph is free.*

Proof (sketch). Let Γ be a graph, $T \subseteq \Gamma$ a spanning tree, and $\{e_\alpha\}$ the set of edges in the complement of T. Suppose also that the edges e_α are equipped with a favored orientation. Then one shows that $\pi_1(\Gamma)$ is free on generators $\{g_\alpha\}$ which are in bijective correspondence to the edges $\{e_\alpha\}$. The generator g_α is the class of the path defined as follows. First, traverse a path inside the tree T from the basepoint to the initial endpoint of the edge e_α, then cross the edge e_α, and finally, return to the basepoint through the tree T. It is easy to show that the g_α's generate $\pi_1(\Gamma)$; that they form a set of free generators is a consequence of Van Kampen's theorem. □

By examining the details of this proof, we can identify a free generating set for the group of closed words, C. Firstly, P is the fundamental group of a bouquet of 2 circles, X. It is easy to identify the covering space corresponding to C, this is simply the skeleton of the square lattice in the plane, call it Y. Of course, there is no canonical choice of spanning tree of Y, nor does there seem to be a "best" choice. We will use the spanning tree consisting of all horizontal edges along the x-axis, and all vertical edges. For the edges in the complement, we choose their favored orientation to be right to left, as shown in Figure 4.6.

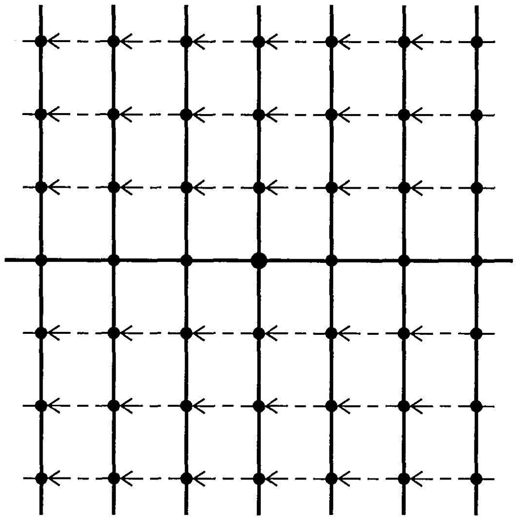

FIGURE 4.6
Spanning tree

Thus we see that C is a free group on the generators

$$b_{ij} = x^{i+1}y^jx^{-1}y^{-j}x^{-i} \qquad \text{for } i \in \mathbf{Z},\ j \in \mathbf{Z} \setminus \{0\}\ .$$

However, it seems better to use a different set of free generators. We will use the following set.

PROPOSITION 4.7. *C is a free group on the generators*

$$c_{ij} = x^iy^jxyx^{-1}y^{-1}y^{-j}x^{-i}\,,$$

over all $i,j \in \mathbf{Z}$.

Proof. We just saw that C is freely generated by the elements $b_{ij} = x^{i+1}y^jx^{-1}y^{-j}x^{-i}$ for $i \in \mathbf{Z}$ and $j \in \mathbf{Z} \setminus \{0\}$. Therefore we can express each c_{ij} uniquely as a word in the b_{ij}'s; the explicit expression is

$$c_{ij} = \begin{cases} b_{i1} & \text{if } j = 0\,, \\ b_{i,-1}^{-1} & \text{if } j = -1\,, \\ b_{ij}^{-1}b_{i,j+1} & \text{otherwise.} \end{cases} \tag{4.8}$$

We can also express the b_{ij}'s in terms of the c_{ij}'s; we get

$$b_{ij} = \begin{cases} c_{i0}c_{i1}c_{i2}\cdots c_{i,j-1} & \text{if } j > 0\,, \\ c_{i,-1}^{-1}c_{i,-2}^{-1}c_{i,-3}^{-1}\cdots c_{ij}^{-1} & \text{if } j < 0\,. \end{cases} \tag{4.9}$$

Therefore the c_{ij}'s generate C. Now we must show freeness. Let G be any group, and for $i,j \in \mathbf{Z}$, let g_{ij} be any element of G. We must show that there is a unique homomorphism $\varphi\colon C \to G$ with $\varphi(c_{ij}) = g_{ij}$ for all i,j.

Expression (4.9) shows that any such φ must satisfy

$$(4.10) \qquad \varphi(b_{ij}) = \begin{cases} g_{i0}g_{i1}g_{i2}\cdots g_{i,j-1} & \text{if } j > 0, \\ g_{i,-1}^{-1}g_{i,-2}^{-1}g_{i,-3}^{-1}\cdots g_{ij}^{-1} & \text{if } j < 0. \end{cases}$$

Since the b_{ij}'s are free generators, there is a unique homomorphism $\varphi\colon C \to G$ satisfying (4.10). Then equation (4.8) shows that indeed $\varphi(c_{ij}) = g_{ij}$ for all i, j. This shows that the c_{ij}'s are free generators, which completes the proof. □

The significance of the c_{ij}'s is that c_{ij} has winding number 1 around the (i,j) cell and has winding number 0 around all other cells. Now we are in a good position to understand the relation between tile homotopy and tile homology.

THEOREM 4.11 (Conway-Lagarias). *The abelianization of the tile homotopy group of $\mathcal{T}$ is its tile homology group.*

Proof. We have $\pi(\mathcal{T})^{\text{ab}} = (C/Z(\mathcal{T}))^{\text{ab}} \cong C^{\text{ab}}/(\text{image of } Z(\mathcal{T}))$. As C is free on the generators c_{ij}, C^{ab} is a free abelian group on the images of these generators. The generators are in bijective correspondence with the cells of the square lattice, so we may think of C^{ab} as the free abelian group on these cells. It remains to determine the image of $Z(\mathcal{T})$ under this identification. $Z(\mathcal{T})$ is generated by all P-conjugates of boundary words of tiles in $\mathcal{T}$. A typical such generator has the form uwu^{-1}, where $u \in P$ is arbitrary, and w is a boundary word of a tile. This corresponds to a closed path (thus an element of C), so it may be written uniquely as a word in the c_{ij}'s. To understand its image in C^{ab}, we need to know the total weight with which each c_{ij} occurs. However, this weight is simply the winding number around cell (i,j), and the winding number is either 1 or 0, depending upon whether the cell occurs in the tile placement or not. Thus the image of uwu^{-1} is the element in the free abelian group on cells that corresponds to this particular tile placement. Now we see that $\pi(\mathcal{T})^{\text{ab}} \cong C^{\text{ab}}/(\text{image of } Z(\mathcal{T})) \cong A/B(\mathcal{T})$, which is the tile homology group. □

Hidden behind the scenes is a topological space, which we now bring to the forefront. Let Y be the skeleton of the square lattice, which we have seen in Figure 4.6. Note that $Y \to X$ is a normal covering map, where X is a bouquet of two circles, and the group of deck transformations is $\mathbf{Z}^2$, acting via translations of the square lattice.

From Y, we build a new space, called $Y(\mathcal{T})$, by sewing in a 2-cell into every possible tile placement. This is a covering space for $X(\mathcal{T})$, which is constructed in a similar way. Namely, we sew in the boundary of a 2-cell along the path corresponding to each boundary word of a tile in $\mathcal{T}$. (Technically, we must sew in a cell for *every possible* boundary word, where all possible base points are considered.) Then $Y(\mathcal{T}) \to X(\mathcal{T})$ is also a normal covering map, again whose group of deck transformations is $\mathbf{Z}^2$ acting via translations of the square lattice. Moreover, the restriction to Y is the covering map $Y \to X$.

The fundamental group of $X(\mathcal{T})$ is the tile path group $P(\mathcal{T})$, and the covering space $Y(\mathcal{T})$ corresponds to the subgroup $\pi(\mathcal{T}) \subseteq P(\mathcal{T})$. The first homology group of $Y(\mathcal{T})$ is the tile homology group, $H(\mathcal{T})$. Thus Theorem 4.11 can be considered as a special case of the Hurewicz Isomorphism Theorem.

5. Strategy for working with tile path groups

We have shown above how to translate tiling problems into problems in finitely presented groups, so we might hope to be able to resolve such questions. Unfortunately, the situation is grim. The so-called *word problem*, as well as many related problems, is known to be unsolvable, which means that no algorithm can answer the question for all possible values of the input.

This is not the end of our story, for we are not trying to solve every word problem. We might hope, however optimistically, that the word problems that arise for us can be solved, whether by hook or by crook. The algorithmic unsolvability of these problems should serve to temper any optimism that we can muster.

The tile homotopy method has been successfully applied in several cases, see [2, Exercise for Experts], [4], [13], [14]. Despite these efforts; results have been found in only a handful of cases. In this section, we give a simple strategy for understanding tile homotopy groups, which allows many new cases to be handled. In view of the difficulty in working with finitely presented groups, we understand that our approach cannot be algorithmic, nor can we expect to be able to apply it in all cases. Nonetheless, we are able to use our strategy to handle numerous new cases.

The tile path group for a finite set $\mathcal{T}$ of prototiles is given by a finitely presented group. We are more interested in the tile homotopy group, which is a subgroup of infinite index. The infiniteness of this index is unfortunate, in light of the following well-known result.