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Kapitel: 1. Deriving Theorem 1 from Theorem 2

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THEOREM 2. *Let* $r-1=\sum_{i\geq 0} a_i p^i$ *and* $s-1=\sum_{i\geq 0} b_i p^i$ *be the respective* p*-adic expansions of* $r-1$ *and* $s-1$*, with* $0\leq a_i, b_i\leq p-1$ *for all* i.

Define the integer k *as the largest index for which* $a_k+b_k\geq p$*, if any exists. Otherwise, that is if* $a_i+b_i\leq p-1$ *for all* $i\geq 0$*, set* $k=-1$.

Then, $\beta_p(r,s)$ *is determined by*

$$\beta_p(r,s)=\left(\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor+\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor+1\right)p^{k+1}. \tag{2}$$

Although the point of Plagne's paper is to stress the relationship of his formula with Additive Number Theory, it is interesting to note that (1) also admits a direct proof using the above Theorem 2.

This is the content of the next section. In Section 2, we provide a simple proof of Theorem 2.

1. DERIVING THEOREM 1 FROM THEOREM 2

It is very easy to understand the relationship of the floor-function $\lfloor\xi\rfloor$, or integral part of ξ, appearing in Theorem 2, with the ceiling-function $\lceil\xi\rceil$, the smallest integer at least as big as ξ, used in formula (1).

The main object of this section will be to locate the minimum over $\ell\geq 0$ of the expression $\left(\left\lceil\frac{r}{p^\ell}\right\rceil+\left\lceil\frac{s}{p^\ell}\right\rceil-1\right)p^\ell$ and to show that this minimum is attained at $\ell=k+1$ with k as defined in Theorem 2.

For every index $\ell\geq 0$, we have

$$0<\frac{1+\sum_{i=0}^{\ell-1} a_i p^i}{p^\ell}\leq\frac{1+\sum_{i=0}^{\ell-1}(p-1)p^i}{p^\ell}=1\,.$$

Since $r=1+\sum_{i\geq 0} a_i p^i$, it follows that

$$\left\lceil\frac{r}{p^\ell}\right\rceil=\sum_{i\geq 0} a_{i+\ell}p^i+1\,.$$

Similarly, we have $0\leq\frac{\sum_{i=0}^{\ell-1} a_i p^i}{p^\ell}\leq\frac{\sum_{i=0}^{\ell-1}(p-1)p^i}{p^\ell}=\frac{p^\ell-1}{p^\ell}<1$, and

$$\left\lfloor\frac{r-1}{p^\ell}\right\rfloor=\sum_{i\geq 0} a_{i+\ell}p^i\,. \tag{3}$$

Hence, $\left\lceil \frac{r}{p^\ell} \right\rceil = \left\lfloor \frac{r-1}{p^\ell} \right\rfloor + 1$.

Applying the same formulas to s, we have $\left\lceil \frac{s}{p^\ell} \right\rceil = \left\lfloor \frac{s-1}{p^\ell} \right\rfloor + 1$. Hence,

$$\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell = \left(\left\lfloor \frac{r-1}{p^\ell} \right\rfloor + \left\lfloor \frac{s-1}{p^\ell} \right\rfloor + 1\right) p^\ell$$

for every ℓ.

It remains to locate the minimum of the expression $\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell$ as a function of ℓ.

If $a_i + b_i \le p-1$ for every $i \ge 0$, then $\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell$ is a weakly increasing function of $\ell \ge 0$. Indeed, the equation

$$\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 = \sum_{i \ge 0} (a_{i+\ell} + b_{i+\ell}) p^i + 1$$

yields for $\ell < \ell'$

$$\begin{aligned}
&\left(\left\lceil \frac{r}{p^{\ell'}} \right\rceil + \left\lceil \frac{s}{p^{\ell'}} \right\rceil - 1\right) p^{\ell'} - \left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell \\
&\quad = (1 + \sum_{i \ge 0} (a_{i+\ell'} + b_{i+\ell'}) p^i) p^{\ell'} - (1 + \sum_{i \ge 0} (a_{i+\ell} + b_{i+\ell}) p^i) p^\ell \\
&\quad = p^{\ell'} - p^\ell - \sum_{\ell \le i < \ell'} (a_i + b_i) p^i \ge p^{\ell'} - p^\ell - \sum_{\ell \le i < \ell'} (p-1) p^i = 0 .
\end{aligned}$$

Thus, in the case where $k = -1$, the minimum of $\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell$ is attained at $\ell = 0$ and $\min_{\ell \ge 0} \left\{\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell\right\} = r+s-1$, as desired.

If there exists an index $k \ge 0$ such that $a_k + b_k \ge p$ and $0 \le a_i + b_i \le p-1$ for $k < i$, then the above calculation shows that $(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1) p^\ell$ is a weakly increasing function of ℓ for $k+1 \le \ell$.

On the other hand, for $\ell \le k$, we have

$$\begin{aligned}
&\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell - \left(\left\lceil \frac{r}{p^{k+1}} \right\rceil + \left\lceil \frac{s}{p^{k+1}} \right\rceil - 1\right) p^{k+1} \\
&\quad = p^\ell - p^{k+1} + \sum_{\ell \le i \le k} (a_i + b_i) p^i \ge p^\ell - p^{k+1} + p^{k+1} = p^\ell > 0 .
\end{aligned}$$

Therefore, even though the function $\left(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1\right) p^\ell$ need not be monotonously decreasing in the interval $0 \le \ell \le k$, and it actually is not in general, it still does take its minimum at $\ell = k+1$.

Consequently, in both cases $k=-1$ and $k\geq 0$, we have

$$\min_{\ell\geq 0}\left(\left\lceil\frac{r}{p^\ell}\right\rceil+\left\lceil\frac{r}{p^\ell}\right\rceil-1\right)p^\ell=\left(\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor+\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor+1\right)p^{k+1}.$$

Now, Theorem 2 tells us that

$$\left(\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor+\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor+1\right)p^{k+1}=\beta_p(r,s),$$

and Theorem 1 follows.

2. Proof of Theorem 2

As noted in equation (3) of Section 1, $\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor=\sum_{i\geq k+1}a_ip^{i-(k+1)}$. Similarly, $\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor=\sum_{i\geq k+1}b_ip^{i-(k+1)}$.

By definition of k, we have $a_i+b_i\leq p-1$ for $i\geq k+1$ and thus the right hand side of the equation

$$\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor+\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor=\sum_{i\geq k+1}(a_i+b_i)p^{i-(k+1)}$$

is the p-adic expansion of the left hand side.

For the purpose of the proof of Theorem 2, set

$$w=\left(\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor+\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor\right)p^{k+1}=\sum_{i\geq k+1}(a_i+b_i)p^i. \tag{4}$$

We proceed to show that $w+p^{k+1}$ is the smallest integer n such that $(x+y)^n$ belongs to the ideal $(x^r,y^s)=x^r\mathbf{F}_p[x,y]+y^s\mathbf{F}_p[x,y]$ in the polynomial ring $\mathbf{F}_p[x,y]$. That is $w+p^{k+1}=\beta_p(r,s)$.

We first calculate $(x+y)^w$ in the quotient algebra of $\mathbf{F}_p[x,y]$ modulo (x^r,y^s). We have from (4)

$$(x+y)^w=\prod_{i\geq k+1}\sum_{c_i=0}^{a_i+b_i}\binom{a_i+b_i}{c_i}x^{c_ip^i}y^{(a_i+b_i-c_i)p^i}.$$

We claim that

$$(x+y)^w\equiv\prod_{i\geq k+1}\binom{a_i+b_i}{a_i}x^{a_ip^i}y^{b_ip^i}=\prod_{i\geq k+1}\binom{a_i+b_i}{a_i}x^uy^v, \tag{5}$$

modulo (x^r,y^s), where $u=\sum_{i\geq k+1}a_ip^i$ and $v=\sum_{i\geq k+1}b_ip^i$.