

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 49 (2003)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** NOTE ON THE HOPF-STIEFEL FUNCTION  
**Autor:** Eliahou, Shalom / Kervaire, Michel  
**Kapitel:** 1. Deriving Theorem 1 from Theorem 2  
**DOI:** <https://doi.org/10.5169/seals-66683>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 15.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

**THEOREM 2.** Let  $r - 1 = \sum_{i \geq 0} a_i p^i$  and  $s - 1 = \sum_{i \geq 0} b_i p^i$  be the respective  $p$ -adic expansions of  $r - 1$  and  $s - 1$ , with  $0 \leq a_i, b_i \leq p - 1$  for all  $i$ .

Define the integer  $k$  as the largest index for which  $a_k + b_k \geq p$ , if any exists. Otherwise, that is if  $a_i + b_i \leq p - 1$  for all  $i \geq 0$ , set  $k = -1$ .

Then,  $\beta_p(r, s)$  is determined by

$$(2) \quad \beta_p(r, s) = \left( \left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1}.$$

Although the point of Plagne's paper is to stress the relationship of his formula with Additive Number Theory, it is interesting to note that (1) also admits a direct proof using the above Theorem 2.

This is the content of the next section. In Section 2, we provide a simple proof of Theorem 2.

## 1. DERIVING THEOREM 1 FROM THEOREM 2

It is very easy to understand the relationship of the floor-function  $\lfloor \xi \rfloor$ , or integral part of  $\xi$ , appearing in Theorem 2, with the ceiling-function  $\lceil \xi \rceil$ , the smallest integer at least as big as  $\xi$ , used in formula (1).

The main object of this section will be to locate the minimum over  $\ell \geq 0$  of the expression  $\left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell$  and to show that this minimum is attained at  $\ell = k + 1$  with  $k$  as defined in Theorem 2.

For every index  $\ell \geq 0$ , we have

$$0 < \frac{1 + \sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{1 + \sum_{i=0}^{\ell-1} (p-1)p^i}{p^\ell} = 1. \quad .$$

Since  $r = 1 + \sum_{i \geq 0} a_i p^i$ , it follows that

$$\left\lceil \frac{r}{p^\ell} \right\rceil = \sum_{i \geq 0} a_{i+\ell} p^i + 1.$$

Similarly, we have  $0 \leq \frac{\sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{\sum_{i=0}^{\ell-1} (p-1)p^i}{p^\ell} = \frac{p^\ell - 1}{p^\ell} < 1$ , and

$$(3) \quad \left\lfloor \frac{r-1}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i.$$

Hence,  $\left\lceil \frac{r}{p^\ell} \right\rceil = \left\lfloor \frac{r-1}{p^\ell} \right\rfloor + 1$ .

Applying the same formulas to  $s$ , we have  $\left\lceil \frac{s}{p^\ell} \right\rceil = \left\lfloor \frac{s-1}{p^\ell} \right\rfloor + 1$ . Hence,

$$\left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell = \left( \left\lfloor \frac{r-1}{p^\ell} \right\rfloor + \left\lfloor \frac{s-1}{p^\ell} \right\rfloor + 1 \right) p^\ell$$

for every  $\ell$ .

It remains to locate the minimum of the expression  $\left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell$  as a function of  $\ell$ .

If  $a_i + b_i \leq p-1$  for every  $i \geq 0$ , then  $\left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell$  is a weakly increasing function of  $\ell \geq 0$ . Indeed, the equation

$$\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 = \sum_{i \geq 0} (a_{i+\ell} + b_{i+\ell}) p^i + 1$$

yields for  $\ell < \ell'$

$$\begin{aligned} & \left( \left\lceil \frac{r}{p^{\ell'}} \right\rceil + \left\lceil \frac{s}{p^{\ell'}} \right\rceil - 1 \right) p^{\ell'} - \left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell \\ &= (1 + \sum_{i \geq 0} (a_{i+\ell'} + b_{i+\ell'}) p^i) p^{\ell'} - (1 + \sum_{i \geq 0} (a_{i+\ell} + b_{i+\ell}) p^i) p^\ell \\ &= p^{\ell'} - p^\ell - \sum_{\ell \leq i < \ell'} (a_i + b_i) p^i \geq p^{\ell'} - p^\ell - \sum_{\ell \leq i < \ell'} (p-1) p^i = 0. \end{aligned}$$

Thus, in the case where  $k = -1$ , the minimum of  $\left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell$  is attained at  $\ell = 0$  and  $\min_{\ell \geq 0} \left\{ \left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell \right\} = r+s-1$ , as desired.

If there exists an index  $k \geq 0$  such that  $a_k + b_k \geq p$  and  $0 \leq a_i + b_i \leq p-1$  for  $k < i$ , then the above calculation shows that  $(\left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1) p^\ell$  is a weakly increasing function of  $\ell$  for  $k+1 \leq \ell$ .

On the other hand, for  $\ell \leq k$ , we have

$$\begin{aligned} & \left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell - \left( \left\lceil \frac{r}{p^{k+1}} \right\rceil + \left\lceil \frac{s}{p^{k+1}} \right\rceil - 1 \right) p^{k+1} \\ &= p^\ell - p^{k+1} + \sum_{\ell \leq i \leq k} (a_i + b_i) p^i \geq p^\ell - p^{k+1} + p^{k+1} = p^\ell > 0. \end{aligned}$$

Therefore, even though the function  $\left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{s}{p^\ell} \right\rceil - 1 \right) p^\ell$  need not be monotonously decreasing in the interval  $0 \leq \ell \leq k$ , and it actually is not in general, it still does take its minimum at  $\ell = k+1$ .

Consequently, in both cases  $k = -1$  and  $k \geq 0$ , we have

$$\min_{\ell \geq 0} \left( \left\lceil \frac{r}{p^\ell} \right\rceil + \left\lceil \frac{r}{p^\ell} \right\rceil - 1 \right) p^\ell = \left( \left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1}.$$

Now, Theorem 2 tells us that

$$\left( \left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1} = \beta_p(r, s),$$

and Theorem 1 follows.

## 2. PROOF OF THEOREM 2

As noted in equation (3) of Section 1,  $\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor = \sum_{i \geq k+1} a_i p^{i-(k+1)}$ . Similarly,  $\left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor = \sum_{i \geq k+1} b_i p^{i-(k+1)}$ .

By definition of  $k$ , we have  $a_i + b_i \leq p - 1$  for  $i \geq k + 1$  and thus the right hand side of the equation

$$\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor = \sum_{i \geq k+1} (a_i + b_i) p^{i-(k+1)}$$

is the  $p$ -adic expansion of the left hand side.

For the purpose of the proof of Theorem 2, set

$$(4) \quad w = \left( \left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor \right) p^{k+1} = \sum_{i \geq k+1} (a_i + b_i) p^i.$$

We proceed to show that  $w + p^{k+1}$  is the smallest integer  $n$  such that  $(x+y)^n$  belongs to the ideal  $(x^r, y^s) = x^r \mathbf{F}_p[x, y] + y^s \mathbf{F}_p[x, y]$  in the polynomial ring  $\mathbf{F}_p[x, y]$ . That is  $w + p^{k+1} = \beta_p(r, s)$ .

We first calculate  $(x+y)^w$  in the quotient algebra of  $\mathbf{F}_p[x, y]$  modulo  $(x^r, y^s)$ . We have from (4)

$$(x+y)^w = \prod_{i \geq k+1} \sum_{c_i=0}^{a_i+b_i} \binom{a_i+b_i}{c_i} x^{c_i p^i} y^{(a_i+b_i-c_i)p^i}.$$

We claim that

$$(5) \quad (x+y)^w \equiv \prod_{i \geq k+1} \binom{a_i+b_i}{a_i} x^{a_i p^i} y^{b_i p^i} = \prod_{i \geq k+1} \binom{a_i+b_i}{a_i} x^u y^v,$$

modulo  $(x^r, y^s)$ , where  $u = \sum_{i \geq k+1} a_i p^i$  and  $v = \sum_{i \geq k+1} b_i p^i$ .