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A NOTE ON THE HOPF-STIEFEL FUNCTION

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INTRODUCTION

In the preceding paper of this volume [P], Alain Plagne gives a formula for the (generalized) Hopf-Stiefel function β_p .

Given a prime number p , and two positive integers r, s , recall that $\beta_p(r, s)$ is defined as the smallest integer n such that $(x + y)^n \in (x^r, y^s)$, where (x^r, y^s) is the ideal generated by x^r and y^s in the polynomial ring $\mathbb{F}_p[x, y]$.

Plagne's theorem reads

THEOREM 1. *Let r, s be positive integers, then $\beta_p(r, s)$ is given by the formula*

$$(1) \quad \beta_p(r, s) = \min_{t \in \mathbb{N}} \left(\left\lceil \frac{r}{p^t} \right\rceil + \left\lceil \frac{s}{p^t} \right\rceil - 1 \right) p^t.$$

In [P], this formula is derived as a corollary of a theorem on Additive Number Theory, Theorem 4, which is the main result of the paper.

Here, we give another proof of Theorem 1 using a purely arithmetical argument.

Recall from [EK, p. 22], where $\beta_p(r, s)$ was introduced, that this function can be described in terms of the p -adic expansions of $r - 1$ and $s - 1$ as follows.

*) During the preparation of this paper, the first author has partially benefited from a research contract with the Fonds National Suisse pour la Recherche Scientifique.

THEOREM 2. Let $r - 1 = \sum_{i \geq 0} a_i p^i$ and $s - 1 = \sum_{i \geq 0} b_i p^i$ be the respective p -adic expansions of $r - 1$ and $s - 1$, with $0 \leq a_i, b_i \leq p - 1$ for all i .

Define the integer k as the largest index for which $a_k + b_k \geq p$, if any exists. Otherwise, that is if $a_i + b_i \leq p - 1$ for all $i \geq 0$, set $k = -1$.

Then, $\beta_p(r, s)$ is determined by

$$(2) \quad \beta_p(r, s) = \left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1}.$$

Although the point of Plagne's paper is to stress the relationship of his formula with Additive Number Theory, it is interesting to note that (1) also admits a direct proof using the above Theorem 2.

This is the content of the next section. In Section 2, we provide a simple proof of Theorem 2.

1. DERIVING THEOREM 1 FROM THEOREM 2

It is very easy to understand the relationship of the floor-function $\lfloor \xi \rfloor$, or integral part of ξ , appearing in Theorem 2, with the ceiling-function $\lceil \xi \rceil$, the smallest integer at least as big as ξ , used in formula (1).

The main object of this section will be to locate the minimum over $\ell \geq 0$ of the expression $\left(\left\lfloor \frac{r}{p^\ell} \right\rfloor + \left\lfloor \frac{s}{p^\ell} \right\rfloor - 1 \right) p^\ell$ and to show that this minimum is attained at $\ell = k + 1$ with k as defined in Theorem 2.

For every index $\ell \geq 0$, we have

$$0 < \frac{1 + \sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{1 + \sum_{i=0}^{\ell-1} (p-1) p^i}{p^\ell} = 1.$$

Since $r = 1 + \sum_{i \geq 0} a_i p^i$, it follows that

$$\left\lfloor \frac{r}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i + 1.$$

Similarly, we have $0 \leq \frac{\sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{\sum_{i=0}^{\ell-1} (p-1) p^i}{p^\ell} = \frac{p^\ell - 1}{p^\ell} < 1$, and

$$(3) \quad \left\lfloor \frac{r-1}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i.$$