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A NOTE ON THE HOPF-STIEFEL FUNCTION

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INTRODUCTION

In the preceding paper of this volume [P], Alain Plagne gives a formula for the (generalized) Hopf-Stiefel function β_p.

Given a prime number p, and two positive integers r, s, recall that $\beta_p(r,s)$ is defined as the smallest integer n such that $(x+y)^n \in (x^r, y^s)$, where (x^r, y^s) is the ideal generated by x^r and y^s in the polynomial ring $\mathbf{F}_p[x,y]$.

Plagne's theorem reads

THEOREM 1. *Let r, s be positive integers, then $\beta_p(r,s)$ is given by the formula*

$$\beta_p(r,s) = \min_{t \in \mathbf{N}} \left(\left\lceil \frac{r}{p^t} \right\rceil + \left\lceil \frac{s}{p^t} \right\rceil - 1 \right) p^t . \tag{1}$$

In [P], this formula is derived as a corollary of a theorem on Additive Number Theory, Theorem 4, which is the main result of the paper.

Here, we give another proof of Theorem 1 using a purely arithmetical argument.

Recall from [EK, p. 22], where $\beta_p(r,s)$ was introduced, that this function can be described in terms of the p-adic expansions of $r-1$ and $s-1$ as follows.

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THEOREM 2. *Let* $r-1=\sum_{i\geq 0} a_i p^i$ *and* $s-1=\sum_{i\geq 0} b_i p^i$ *be the respective* p*-adic expansions of* $r-1$ *and* $s-1$*, with* $0\leq a_i, b_i \leq p-1$ *for all* i.

Define the integer k *as the largest index for which* $a_k+b_k\geq p$*, if any exists. Otherwise, that is if* $a_i+b_i\leq p-1$ *for all* $i\geq 0$*, set* $k=-1$.

Then, $\beta_p(r,s)$ *is determined by*

$$\beta_p(r,s)=\left(\left\lfloor\frac{r-1}{p^{k+1}}\right\rfloor+\left\lfloor\frac{s-1}{p^{k+1}}\right\rfloor+1\right)p^{k+1}. \tag{2}$$

Although the point of Plagne's paper is to stress the relationship of his formula with Additive Number Theory, it is interesting to note that (1) also admits a direct proof using the above Theorem 2.

This is the content of the next section. In Section 2, we provide a simple proof of Theorem 2.

1. DERIVING THEOREM 1 FROM THEOREM 2

It is very easy to understand the relationship of the floor-function $\lfloor\xi\rfloor$, or integral part of ξ, appearing in Theorem 2, with the ceiling-function $\lceil\xi\rceil$, the smallest integer at least as big as ξ, used in formula (1).

The main object of this section will be to locate the minimum over $\ell\geq 0$ of the expression $\left(\left\lceil\frac{r}{p^\ell}\right\rceil+\left\lceil\frac{s}{p^\ell}\right\rceil-1\right)p^\ell$ and to show that this minimum is attained at $\ell=k+1$ with k as defined in Theorem 2.

For every index $\ell\geq 0$, we have

$$0<\frac{1+\sum_{i=0}^{\ell-1}a_ip^i}{p^\ell}\leq\frac{1+\sum_{i=0}^{\ell-1}(p-1)p^i}{p^\ell}=1.$$

Since $r=1+\sum_{i\geq 0}a_ip^i$, it follows that

$$\left\lceil\frac{r}{p^\ell}\right\rceil=\sum_{i\geq 0}a_{i+\ell}p^i+1.$$

Similarly, we have $0\leq\frac{\sum_{i=0}^{\ell-1}a_ip^i}{p^\ell}\leq\frac{\sum_{i=0}^{\ell-1}(p-1)p^i}{p^\ell}=\frac{p^\ell-1}{p^\ell}<1$, and

$$\left\lfloor\frac{r-1}{p^\ell}\right\rfloor=\sum_{i\geq 0}a_{i+\ell}p^i. \tag{3}$$