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A NOTE ON THE HOPF-STIEFEL FUNCTION

by Shalom ELIAHOU*) and Michel KERVAIRE

INTRODUCTION

In the preceding paper of this volume [P], Alain Plagne gives a formula for the (generalized) Hopf-Stiefel function β_p .

Given a prime number p , and two positive integers r, s , recall that $\beta_p(r, s)$ is defined as the smallest integer n such that $(x + y)^n \in (x^r, y^s)$, where (x^r, y^s) is the ideal generated by x^r and y^s in the polynomial ring $\mathbb{F}_p[x, y]$.

Plagne's theorem reads

THEOREM 1. *Let r, s be positive integers, then $\beta_p(r, s)$ is given by the formula*

$$(1) \quad \beta_p(r, s) = \min_{t \in \mathbb{N}} \left(\left\lceil \frac{r}{p^t} \right\rceil + \left\lceil \frac{s}{p^t} \right\rceil - 1 \right) p^t.$$

In [P], this formula is derived as a corollary of a theorem on Additive Number Theory, Theorem 4, which is the main result of the paper.

Here, we give another proof of Theorem 1 using a purely arithmetical argument.

Recall from [EK, p. 22], where $\beta_p(r, s)$ was introduced, that this function can be described in terms of the p -adic expansions of $r - 1$ and $s - 1$ as follows.

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THEOREM 2. Let $r - 1 = \sum_{i \geq 0} a_i p^i$ and $s - 1 = \sum_{i \geq 0} b_i p^i$ be the respective p -adic expansions of $r - 1$ and $s - 1$, with $0 \leq a_i, b_i \leq p - 1$ for all i .

Define the integer k as the largest index for which $a_k + b_k \geq p$, if any exists. Otherwise, that is if $a_i + b_i \leq p - 1$ for all $i \geq 0$, set $k = -1$.

Then, $\beta_p(r, s)$ is determined by

$$(2) \quad \beta_p(r, s) = \left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1}.$$

Although the point of Plagne's paper is to stress the relationship of his formula with Additive Number Theory, it is interesting to note that (1) also admits a direct proof using the above Theorem 2.

This is the content of the next section. In Section 2, we provide a simple proof of Theorem 2.

1. DERIVING THEOREM 1 FROM THEOREM 2

It is very easy to understand the relationship of the floor-function $\lfloor \xi \rfloor$, or integral part of ξ , appearing in Theorem 2, with the ceiling-function $\lceil \xi \rceil$, the smallest integer at least as big as ξ , used in formula (1).

The main object of this section will be to locate the minimum over $\ell \geq 0$ of the expression $\left(\left\lfloor \frac{r}{p^\ell} \right\rfloor + \left\lfloor \frac{s}{p^\ell} \right\rfloor - 1 \right) p^\ell$ and to show that this minimum is attained at $\ell = k + 1$ with k as defined in Theorem 2.

For every index $\ell \geq 0$, we have

$$0 < \frac{1 + \sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{1 + \sum_{i=0}^{\ell-1} (p-1) p^i}{p^\ell} = 1.$$

Since $r = 1 + \sum_{i \geq 0} a_i p^i$, it follows that

$$\left\lfloor \frac{r}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i + 1.$$

Similarly, we have $0 \leq \frac{\sum_{i=0}^{\ell-1} a_i p^i}{p^\ell} \leq \frac{\sum_{i=0}^{\ell-1} (p-1) p^i}{p^\ell} = \frac{p^\ell - 1}{p^\ell} < 1$, and

$$(3) \quad \left\lfloor \frac{r-1}{p^\ell} \right\rfloor = \sum_{i \geq 0} a_{i+\ell} p^i.$$

Hence, $\left[\frac{r}{p^\ell} \right] = \left[\frac{r-1}{p^\ell} \right] + 1.$

Applying the same formulas to s , we have $\left[\frac{s}{p^\ell} \right] = \left[\frac{s-1}{p^\ell} \right] + 1.$ Hence,

$$\left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell = \left(\left[\frac{r-1}{p^\ell} \right] + \left[\frac{s-1}{p^\ell} \right] + 1 \right) p^\ell$$

for every $\ell.$

It remains to locate the minimum of the expression $\left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell$ as a function of $\ell.$

If $a_i + b_i \leq p - 1$ for every $i \geq 0,$ then $\left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell$ is a weakly increasing function of $\ell \geq 0.$ Indeed, the equation

$$\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 = \sum_{i \geq 0} (a_{i+\ell} + b_{i+\ell}) p^i + 1$$

yields for $\ell < \ell'$

$$\begin{aligned} & \left(\left[\frac{r}{p^{\ell'}} \right] + \left[\frac{s}{p^{\ell'}} \right] - 1 \right) p^{\ell'} - \left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell \\ &= \left(1 + \sum_{i \geq 0} (a_{i+\ell'} + b_{i+\ell'}) p^i \right) p^{\ell'} - \left(1 + \sum_{i \geq 0} (a_{i+\ell} + b_{i+\ell}) p^i \right) p^\ell \\ &= p^{\ell'} - p^\ell - \sum_{\ell \leq i < \ell'} (a_i + b_i) p^i \geq p^{\ell'} - p^\ell - \sum_{\ell \leq i < \ell'} (p-1) p^i = 0. \end{aligned}$$

Thus, in the case where $k = -1,$ the minimum of $\left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell$ is attained at $\ell = 0$ and $\min_{\ell \geq 0} \left\{ \left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell \right\} = r + s - 1,$ as desired.

If there exists an index $k \geq 0$ such that $a_k + b_k \geq p$ and $0 \leq a_i + b_i \leq p - 1$ for $k < i,$ then the above calculation shows that $\left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell$ is a weakly increasing function of ℓ for $k + 1 \leq \ell.$

On the other hand, for $\ell \leq k,$ we have

$$\begin{aligned} & \left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell - \left(\left[\frac{r}{p^{k+1}} \right] + \left[\frac{s}{p^{k+1}} \right] - 1 \right) p^{k+1} \\ &= p^\ell - p^{k+1} + \sum_{\ell \leq i \leq k} (a_i + b_i) p^i \geq p^\ell - p^{k+1} + p^{k+1} = p^\ell > 0. \end{aligned}$$

Therefore, even though the function $\left(\left[\frac{r}{p^\ell} \right] + \left[\frac{s}{p^\ell} \right] - 1 \right) p^\ell$ need not be monotonously decreasing in the interval $0 \leq \ell \leq k,$ and it actually is not in general, it still does take its minimum at $\ell = k + 1.$

Consequently, in both cases $k = -1$ and $k \geq 0$, we have

$$\min_{\ell \geq 0} \left(\left\lfloor \frac{r}{p^\ell} \right\rfloor + \left\lfloor \frac{r}{p^\ell} \right\rfloor - 1 \right) p^\ell = \left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1}.$$

Now, Theorem 2 tells us that

$$\left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1} = \beta_p(r, s),$$

and Theorem 1 follows.

2. PROOF OF THEOREM 2

As noted in equation (3) of Section 1, $\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor = \sum_{i \geq k+1} a_i p^{i-(k+1)}$.

Similarly, $\left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor = \sum_{i \geq k+1} b_i p^{i-(k+1)}$.

By definition of k , we have $a_i + b_i \leq p - 1$ for $i \geq k + 1$ and thus the right hand side of the equation

$$\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor = \sum_{i \geq k+1} (a_i + b_i) p^{i-(k+1)}$$

is the p -adic expansion of the left hand side.

For the purpose of the proof of Theorem 2, set

$$(4) \quad w = \left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor \right) p^{k+1} = \sum_{i \geq k+1} (a_i + b_i) p^i.$$

We proceed to show that $w + p^{k+1}$ is the smallest integer n such that $(x+y)^n$ belongs to the ideal $(x^r, y^s) = x^r \mathbf{F}_p[x, y] + y^s \mathbf{F}_p[x, y]$ in the polynomial ring $\mathbf{F}_p[x, y]$. That is $w + p^{k+1} = \beta_p(r, s)$.

We first calculate $(x+y)^w$ in the quotient algebra of $\mathbf{F}_p[x, y]$ modulo (x^r, y^s) . We have from (4)

$$(x+y)^w = \prod_{i \geq k+1} \sum_{c_i=0}^{a_i+b_i} \binom{a_i+b_i}{c_i} x^{c_i p^i} y^{(a_i+b_i-c_i)p^i}.$$

We claim that

$$(5) \quad (x+y)^w \equiv \prod_{i \geq k+1} \binom{a_i+b_i}{a_i} x^{a_i p^i} y^{b_i p^i} = \prod_{i \geq k+1} \binom{a_i+b_i}{a_i} x^u y^v,$$

modulo (x^r, y^s) , where $u = \sum_{i \geq k+1} a_i p^i$ and $v = \sum_{i \geq k+1} b_i p^i$.

Indeed, since $a_i + b_i \leq p - 1$ for $i \geq k + 1$ by definition of k , the expressions $c = \sum_{i \geq k+1} c_i p^i$ and $d = \sum_{i \geq k+1} (a_i + b_i - c_i) p^i$ are the p -adic expansions of c and d respectively.

If for a given c , there is an index $i \geq k + 1$ for which c_i is not equal to a_i , denote by ℓ the largest i such that $c_\ell \neq a_\ell$.

If $c_\ell < a_\ell$ and $c_i = a_i$ for $i \geq \ell + 1$, this implies $a_\ell + b_\ell - c_\ell > b_\ell$ and $a_i + b_i - c_i = b_i$ for $i \geq \ell + 1$. Therefore we have

$$d \geq \sum_{k+1 \leq i \leq \ell-1} (a_i + b_i - c_i) p^i + p^\ell + \sum_{i \geq \ell} b_i p^i \geq p^\ell + \sum_{i \geq \ell} b_i p^i \geq s.$$

Thus in this case the monomial $x^c y^d$ belongs to the ideal (x^r, y^s) .

If, on the contrary, $c_\ell > a_\ell$ and $c_i = a_i$ for $i \geq \ell + 1$, this implies

$$c = \sum_{i \geq k+1} c_i p^i \geq \sum_{k+1 \leq i \leq \ell-1} c_i p^i + p^\ell + \sum_{i \geq \ell} a_i p^i \geq r.$$

Thus $(x + y)^w$ is indeed given by formula (5) modulo (x^r, y^s) .

Now, observe that the product of binomial coefficients $\gamma = \prod_{i \geq k+1} \binom{a_i + b_i}{a_i}$ is non-zero in \mathbf{F}_p and we can write $(x + y)^w \equiv \gamma \cdot x^u y^v$ modulo (x^r, y^s) .

It is now easy to finish up the proof of the theorem:

- $(x + y)^{p^{k+1} + w} = (x^{p^{k+1}} + y^{p^{k+1}})(x + y)^w \equiv \gamma \cdot (x^{p^{k+1} + u} y^v + x^u y^{p^{k+1} + v}).$

However, $p^{k+1} + u = 1 + \sum_{i=0}^k (p - 1) p^i + \sum_{i \geq k+1} a_i p^i \geq 1 + (r - 1) = r$. Similarly, $p^{k+1} + v \geq s$.

Summarizing, $(x + y)^{p^{k+1} + w} \in (x^r, y^s)$ and thus

$$\left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1} \geq \beta_p(r, s).$$

- $(x + y)^{w + p^{k+1} - 1} = \gamma \cdot \left(\sum_{j=0}^{p^{k+1} - 1} (-1)^j x^j y^{p^{k+1} - j - 1} \right) x^u y^v,$

using $(x + y)^{p^{k+1} - 1} = \frac{(x^{p^{k+1}} + y^{p^{k+1}})}{x + y} = \sum_{j=0}^{p^{k+1} - 1} (-1)^j x^j y^{p^{k+1} - j - 1}$ in $\mathbf{F}_p[x, y]$.

It is immediate to see that, calculating modulo (x^r, y^s) , and with the notation $u_0 = \sum_{i=0}^k a_i$ and $v_0 = \sum_{i=0}^k b_i$, we can restrict the summation over j to the interval $p^{k+1} - 1 - v_0 \leq j \leq u_0$:

$$(x + y)^{w + p^{k+1} - 1} \equiv \gamma \cdot \left(\sum_{j=p^{k+1} - 1 - v_0}^{j=u_0} (-1)^j x^j y^{p^{k+1} - j - 1} \right) x^u y^v.$$

Moreover, the monomials appearing on the right hand side are distinct, have non-zero coefficient $\pm\gamma$ and form a non-empty subset of an \mathbf{F}_p -basis of $\mathbf{F}_p[x, y]/(x^r, y^s)$. Indeed, on the one hand, $p^{k+1} - 1 - v_0 \leq u_0$ in view of the inequalities

$$u_0 + v_0 = \sum_{i=0}^k (a_i + b_i)p^i \geq (a_k + b_k)p^k \text{ and } a_k + b_k \geq p,$$

and on the other hand $j+u \leq u_0+u = r-1$ and $p^{k+1}-j-1+v \leq v_0+v = s-1$. If $k = -1$, then $u_0 = v_0 = 0$ and the above conclusion still holds.

Summarizing:

$$\left(\left\lfloor \frac{r-1}{p^{k+1}} \right\rfloor + \left\lfloor \frac{s-1}{p^{k+1}} \right\rfloor + 1 \right) p^{k+1} = \beta_p(r, s),$$

and this completes the proof of Theorem 2.

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