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Artikel: ADDITIVE NUMBER THEORY SHEDS EXTRA LIGHT ON THE HOPF-STIEFEL o FUNCTION
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2.2 The upper bound

Let H be any subgroup of G. Choose $\mathcal{A}_0$ and $\mathcal{B}_0$ in G/H with respective cardinalities $\lceil r/|H|\rceil$ and $\lceil s/|H|\rceil$ and such that

$$|\mathcal{A}_0 + \mathcal{B}_0| = \mu_{G/H}(\lceil r/|H|\rceil, \lceil s/|H|\rceil).$$

Now choose $\mathcal{A}$ of cardinality r and $\mathcal{B}$ of cardinality s in G such that the image of $\mathcal{A}$ (resp. $\mathcal{B}$) by the canonical projection on G/H is included in $\mathcal{A}_0$ (resp. $\mathcal{B}_0$). One has

$$|\mathcal{A} + \mathcal{B}| \leq \mu_{G/H}(\lceil r/|H|\rceil, \lceil s/|H|\rceil)|H|.$$

This proves the first lemma we need.

LEMMA 1. *For any finite Abelian group G*

$$\mu_G(r,s) \leq \min_{H \leq G} \mu_{G/H}(\lceil r/|H|\rceil, \lceil s/|H|\rceil)|H|.$$

The second useful point is synthesized in the next folkloric lemma.

LEMMA 2. *Let G be a finite Abelian group. For any positive integer m, the following two propositions are equivalent*

(i) *m divides* $\exp G$,

(ii) *there exists a subgroup H of G such that G/H is isomorphic to $\mathbf{Z}/m\mathbf{Z}$.*

In the case of a *cyclic* group K, trivial considerations (take two sets with consecutive elements), show that, for any $u, v \leq |K|$,

$$\mu_K(r,s) \leq r + s - 1. \tag{2.2}$$

Using consecutively Lemma 1, inequality (2.2) and Lemma 2 yields the following chain of inequalities:

$$\begin{aligned}
\mu_G(r,s) &\leq \min_{H \leq G} \mu_{G/H}(\lceil r/|H|\rceil, \lceil s/|H|\rceil)|H| \\
&\leq \min_{H \leq G,\ G/H \text{ cyclic}} \mu_{G/H}(\lceil r/|H|\rceil, \lceil s/|H|\rceil)|H| \\
&\leq \min_{H \leq G,\ G/H \text{ cyclic}} \big(\lceil r/|H|\rceil + \lceil s/|H|\rceil - 1\big)\,|H| \\
&= \min_{|G/H| \text{ divides } \exp G} \big(\lceil r/|H|\rceil + \lceil s/|H|\rceil - 1\big)\,|H| \\
&= \min_{f \mid \exp G} \big(\lceil rf/|G|\rceil + \lceil sf/|G|\rceil - 1\big)\,\frac{|G|}{f}.
\end{aligned}$$

The change of variable $d = |G|/f$ yields a parameter d subject to the two restrictions $\frac{|G|}{\exp G} \mid d$ and $d \mid |G|$; this proves the upper bound in Theorem 4.