With these results, we know the behaviour of μ_G at the two endpoints of the spectrum (cyclic groups and groups of prime exponent). What now remains to be done is to fill the gap between the upper bound and the lower bound for general finite Abelian groups.

2. Proof of Theorem 4

Let G be any given finite Abelian group and let $1 \leq r, s \leq |G|$.

2.1 The lower bound

If $\mu_G(r,s) \geq r+s-1$, the result is immediate (take $d=1$). We may thus assume that

$$\mu_G(r,s) \leq r+s-1\,. \tag{2.1}$$

Then, choosing two sets $\mathcal{A}$ and $\mathcal{B}$ in G with respective cardinalities r and s, such that $|\mathcal{A}+\mathcal{B}|$ attains $\mu_G(r,s)$, we get

$$|\mathcal{A}+\mathcal{B}| = \mu_G(r,s) \leq |\mathcal{A}| + |\mathcal{B}| - 1\,.$$

We are in a position to apply Kneser's theorem [9] on the structure of sets with a small sumset. It follows that there exists a subgroup H of G (namely the stabilizer of $\mathcal{A}+\mathcal{B}$) such that

$$|\mathcal{A}+\mathcal{B}| = |\mathcal{A}+H| + |\mathcal{B}+H| - |H|\,.$$

Denoting by $(\mathcal{A}+H)/H$ (resp. $(\mathcal{B}+H)/H$) the H-cosets that $\mathcal{A}$ (resp. $\mathcal{B}$) intersects, we obtain

$$\begin{aligned} |\mathcal{A}+\mathcal{B}| &= \left(\left|\frac{\mathcal{A}+H}{H}\right| + \left|\frac{\mathcal{B}+H}{H}\right| - 1\right)|H| \qquad . \\ &\geq (\lceil r/f \rceil + \lceil s/f \rceil - 1)f \end{aligned}$$

where f denotes the cardinality of H. Since Lagrange's theorem shows that f divides $|G|$, we get

$$|\mathcal{A}+\mathcal{B}| \geq \min_{d \mid |G|} (\lceil r/d \rceil + \lceil s/d \rceil - 1)d\,.$$

From this it follows that, in any case,

$$\mu_G(r,s) \geq \min_{d \mid |G|} (\lceil r/d \rceil + \lceil s/d \rceil - 1)d\,,$$

which is the desired lower bound.