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ADDITIVE NUMBER THEORY SHEDS EXTRA LIGHT ON THE HOPF-STIEFEL $\circ$ FUNCTION

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ABSTRACT. The famous Hopf-Stiefel $\circ$ function appears in several places in mathematics (linear and bilinear algebra, topology, intercalate matrices, ...). However, although the object of much study, this function kept a part of mystery since no simple formula was known for it. We shall derive a simple and practical explicit formula for $\circ$ and more generally for β_p (p arbitrary prime), a generalized function due to Eliahou and Kervaire. The proof relies on a new result in combinatorial group theory which follows from additive number theoretical arguments. It is shown that this last result generalizes earlier ones by Eliahou and Kervaire and by Yuzvinsky.

1. INTRODUCTION

A *composition formula* of size $[r,s,n]$ over some field $\mathbf{F}$ (that we assume to be of characteristic different from 2) is an identity of the form

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = (z_1^2 + \cdots + z_n^2),$$

where $z_1, z_2, \ldots, z_n$ are n bilinear forms in the variables $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_s)$, with coefficients in $\mathbf{F}$. For example, the law of moduli for complex numbers provides the identity

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_2y_1 + x_1y_2)^2,$$

which is a composition formula of size $[2,2,2]$. The law of moduli for quaternions (respectively for octonions) provides a composition formula of size $[4,4,4]$ (respectively $[8,8,8]$) in a similar way. Conversely, Hurwitz's theorem [7] (see also [8]) states that the only possible values of n for which a composition formula of size $[n,n,n]$ exists are $1, 2, 4$ and 8.