

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREES FOUR AND FIVE AND RELATED DIOPHANTINE SYSTEMS
Autor: Choudhry, Ajai
Kapitel: 3. Ideal non-symmetric solutions of the Tarry-Escott problem of degree five
DOI: <https://doi.org/10.5169/seals-66681>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 01.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

We may apply the theorem of Gloden [2, p. 24] to the three-parameter ideal non-symmetric solution obtained above to derive a solution of the system of equations

$$(2.16) \quad \sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r, \quad r = 1, 2, 3, 4, 6,$$

in terms of polynomials of degree six in three parameters. We, however, restrict ourselves to applying this theorem to the simpler solution (2.15), and obtain the following solution of the system of equations (2.16):

$$(2.17) \quad \begin{aligned} a_1 &= 9p^3 + 5p^2 - 3p + 5, & b_1 &= -6p^3 - 10p^2 - 8p, \\ a_2 &= -6p^3 - 15p^2 + 2p - 5, & b_2 &= -6p^3 + 15p^2 + 2p + 5, \\ a_3 &= -6p^3 + 10p^2 - 8p, & b_3 &= 9p^3 - 5p^2 - 3p - 5, \\ a_4 &= 9p^3 - 5p^2 + 7p + 5, & b_4 &= 9p^3 + 5p^2 + 7p - 5, \\ a_5 &= -6p^3 + 5p^2 + 2p - 5, & b_5 &= -6p^3 - 5p^2 + 2p + 5. \end{aligned}$$

When $p = -2$, this leads to the following solution of the system of equations (2.16):

$$(-101)^r + (-41)^r + (-21)^r + 59^r + 104^r = (-91)^r + (-71)^r + 24^r + 29^r + 109^r$$

where $r = 1, 2, 3, 4, 6$.

We note that additional parametric non-symmetric solutions of the Tarry-Escott problem of degree four may be obtained by taking a_i, b_i , as in (2.6), and instead of imposing the condition $a_6 = b_6$, we reduce one term on either side by solving (2.8) together with another condition such as $a_4 = b_6$ or $a_5 = b_6$. Solutions obtained in this manner are of degrees 6, 7 or 8 in terms of three parameters.

3. IDEAL NON-SYMMETRIC SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FIVE

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree five, we have to obtain a solution of the system of equations

$$(3.1) \quad \sum_{i=1}^6 a_i^r = \sum_{i=1}^6 b_i^r, \quad r = 1, 2, 3, 4, 5.$$

We will choose $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ as in (2.6) when (3.1) holds identically for $r = 1, 2, 4$. For $r = 3$, equation (3.1) reduces to (2.8) while for $r = 5$ it reduces to the equation:

$$(3.2) \quad m_1 n_1 (m_1 + n_1) x_1 y_1 (x_1 + y_1) (m_1^2 + m_1 n_1 + n_1^2) (x_1^2 + x_1 y_1 + y_1^2) \\ = m_2 n_2 (m_2 + n_2) x_2 y_2 (x_2 + y_2) (m_2^2 + m_2 n_2 + n_2^2) (x_2^2 + x_2 y_2 + y_2^2).$$

It therefore suffices to solve equation (2.8) together with the following equation:

$$(3.3) \quad (m_1^2 + m_1 n_1 + n_1^2) (x_1^2 + x_1 y_1 + y_1^2) = (m_2^2 + m_2 n_2 + n_2^2) (x_2^2 + x_2 y_2 + y_2^2).$$

We take x_1, y_1, m_1, m_2, n_2 such that

$$(3.4) \quad \begin{aligned} x_1 &= (t^2 + t - 1)x_2, \\ y_1 &= (t + 1)^2 y_2, \\ m_1 &= tx_2 + ty_2, \\ m_2 &= (t^2 + t - 1)x_2 + (t + 1)^2 y_2, \\ n_2 &= (-t^2 - t)n_1. \end{aligned}$$

Substituting these values of x_1, y_1, m_1, m_2, n_2 in (2.8) and solving for n_1 , we get

$$(3.5) \quad n_1 = -\frac{(t^4 + 2t^3 + t^2 - 1)x_2 + (t^4 + 2t^3 + t^2 + t + 1)y_2}{t^3 + t^2 - t - 1},$$

and now (3.4) gives

$$(3.6) \quad n_2 = \frac{(t^5 + 2t^4 + t^3 - t)x_2 + (t^5 + 2t^4 + t^3 + t^2 + t)y_2}{t^2 - 1}.$$

On substituting the values of n_1, n_2 given by (3.5) and (3.6), and the values of x_1, y_1, m_1, m_2 given by (3.4) in equation (3.3), we get the equation:

$$(tx_2 + (t + 1)y_2)((t^2 + t - 1)x_2 + (t + 1)y_2)((t^2 + t - 1)x_2 + (t^2 + t)y_2) \\ \times (t + 2)((2t^5 + 5t^4 + 3t^3 - t^2 - t + 1)x_2 + (t^5 - 6t^3 - 8t^2 - 4t - 1)y_2) = 0.$$

Equating any of the first four factors on the left-hand side of this equation to zero leads either to trivial solutions or to known symmetric solutions of the Tarry-Escott problem of degree five. However, on equating the last factor to zero, we get

$$(3.7) \quad \begin{aligned} x_2 &= t^5 - 6t^3 - 8t^2 - 4t - 1, \\ y_2 &= -(2t^5 + 5t^4 + 3t^3 - t^2 - t + 1), \end{aligned}$$

and now, using the relations (3.4), (3.5) and (3.6), we get the values of $x_1, y_1, m_1, m_2, n_1, n_2$. The values of $x_1, y_1, x_2, y_2, m_1, m_2, n_1, n_2$, may now be substituted in (2.6) to get a non-symmetric solution of the Tarry-Escott problem of degree five. After removing the common factors, this solution may be written as follows:

$$\begin{aligned}
 a_1 &= -3t^{11} - 20t^{10} - 58t^9 - 94t^8 - 106t^7 - 100t^6 - 40t^5 \\
 &\quad + 50t^4 + 50t^3 + 2t^2 - 5t, \\
 a_2 &= 3t^{11} + 16t^{10} + 38t^9 + 71t^8 + 128t^7 + 149t^6 + 56t^5 \\
 &\quad - 37t^4 - 22t^3 + 5t^2 + t - 3, \\
 a_3 &= 4t^{10} + 20t^9 + 23t^8 - 22t^7 - 49t^6 - 16t^5 \\
 &\quad - 13t^4 - 28t^3 - 7t^2 + 4t + 3, \\
 a_4 &= 8t^{10} + 52t^9 + 127t^8 + 148t^7 + 85t^6 + 22t^5 \\
 &\quad + t^4 - 14t^3 - 23t^2 - 4t + 3, \\
 a_5 &= 3t^{11} + 14t^{10} + 16t^9 - 29t^8 - 98t^7 - 89t^6 + 4t^5 + 55t^4 \\
 &\quad + 40t^3 + 13t^2 - 7t - 3, \\
 a_6 &= -3t^{11} - 22t^{10} - 68t^9 - 98t^8 - 50t^7 + 4t^6 - 26t^5 \\
 &\quad - 56t^4 - 26t^3 + 10t^2 + 11t, \\
 b_1 &= 2t^{10} + 13t^9 + 58t^8 + 151t^7 + 190t^6 + 79t^5 \\
 &\quad - 44t^4 - 41t^3 - 2t^2 - t, \\
 b_2 &= -3t^{11} - 19t^{10} - 56t^9 - 116t^8 - 164t^7 - 104t^6 + 34t^5 \\
 &\quad + 76t^4 + 28t^3 + 4t^2 - t - 3, \\
 b_3 &= 3t^{11} + 17t^{10} + 43t^9 + 58t^8 + 13t^7 - 86t^6 - 113t^5 \\
 &\quad - 32t^4 + 13t^3 - 2t^2 + 2t + 3, \\
 b_4 &= 10t^{10} + 59t^9 + 140t^8 + 167t^7 + 98t^6 - t^5 \\
 &\quad - 58t^4 - 31t^3 + 14t^2 + 7t, \\
 b_5 &= -3t^{11} - 23t^{10} - 67t^9 - 88t^8 - 25t^7 + 68t^6 + 71t^5 \\
 &\quad + 14t^4 - 13t^3 - 16t^2 - 2t + 3, \\
 b_6 &= 3t^{11} + 13t^{10} + 8t^9 - 52t^8 - 142t^7 - 166t^6 - 70t^5 \\
 &\quad + 44t^4 + 44t^3 + 2t^2 - 5t - 3.
 \end{aligned}
 \tag{3.8}$$

Here t is an arbitrary rational parameter and integer solutions of (3.1) are obtained by multiplying any rational numerical solution by a suitable constant. The a_i, b_i obtained above satisfy the relation $\sum_{i=1}^6 a_i = \sum_{i=1}^6 b_i = 0$.

It therefore follows from the theorem of Gloden [2, p.24] that these a_i, b_i must also satisfy the relation

$$\sum_{i=1}^6 a_i^7 = \sum_{i=1}^6 b_i^7.$$

This is also verified by direct computation. Hence the a_i, b_i given by (3.8) constitute a solution of the following system of equations:

$$\sum_{i=1}^6 a_i^r = \sum_{i=1}^6 b_i^r, \quad r = 1, 2, 3, 4, 5, 7.$$

As a numerical example, when $t = -3$, we get, after removal of common factors and suitable re-arrangement, the following solution

$$\begin{aligned} (-19323)^r + (-18689)^r + 3117^r + 5111^r + 14212^r + 15572^r \\ = (-20023)^r + (-17828)^r + 1017^r + 9787^r + 10236^r + 16811^r \end{aligned}$$

where $r = 1, 2, 3, 4, 5, 7$.

REFERENCES

- [1] CHOUDHRY, A. Ideal Solutions of the Tarry-Escott problem of degree four and a related diophantine system. *L'Enseignement Math. (2)* 46 (2000), 313–323.
- [2] GLODEN, A. *Mehrgradige Gleichungen*. Noordhoff, Groningen, 1944.

(Reçu le 27 novembre 2002)

Ajai Choudhry

High Commissioner
 High Commission of India
 P.O. Box 439, Mail Processing Centre
 Airport Lama, Berakas
 Bandar Seri Begawan, BB3577
 Brunei
e-mail: ajaic203@yahoo.com