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Artikel: IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREES FOUR AND FIVE AND RELATED DIOPHANTINE SYSTEMS

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Kapitel: 2. Ideal non-symmetric solutions of the Tarry-Escott problem of degree four

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$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r\,, \quad r = 1, 2, \ldots, k, k+2 \tag{1.1}$$

by applying a theorem of Gloden [2, p. 24]. Applying this procedure to the non-symmetric ideal solutions of degrees four and five obtained in this paper, we get parametric solutions of (1.1) when $k = 4$ or $k = 5$.

2. Ideal non-symmetric solutions of the Tarry-Escott problem of degree four

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree four, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r\,, \qquad r = 1, 2, 3, 4. \tag{2.1}$$

We first observe that the system of equations

$$X_1^r + X_2^r + X_3^r = Y_1^r + Y_2^3 + Y_3^r\,, \qquad r = 1, 2, 4, \tag{2.2}$$

reduces to

$$X_1^2 + X_1X_2 + X_2^2 = Y_1^2 + Y_1Y_2 + Y_2^2\,, \tag{2.3}$$

if we take $X_3 = -X_1 - X_2$ and $Y_3 = -Y_1 - Y_2$. A solution of (2.3) in terms of arbitrary parameters m, n, x, y, is given by

$$\begin{aligned} X_1 &= (m+2n)x + (-m+n)y\,, \\ X_2 &= (-2m-n)x + (-m-2n)y\,, \\ Y_1 &= (m-n)x + (-m-2n)y\,, \\ Y_2 &= (-2m-n)x + (-m+n)y\,, \end{aligned} \tag{2.4}$$

and we now get

$$\begin{aligned} X_3 &= (m-n)x + (2m+n)y\,, \\ Y_3 &= (m+2n)x + (2m+n)y\,. \end{aligned} \tag{2.5}$$

It follows from this solution of the system of equations (2.2) that if we take

$$
\begin{aligned}
a_1 &= (m_1 + 2n_1)x_1 + (-m_1 + n_1)y_1\,, \\
a_2 &= (-2m_1 - n_1)x_1 + (-m_1 - 2n_1)y_1\,, \\
a_3 &= (m_1 - n_1)x_1 + (2m_1 + n_1)y_1\,, \\
a_4 &= (m_2 + 2n_2)x_2 + (2m_2 + n_2)y_2\,, \\
a_5 &= (-2m_2 - n_2)x_2 + (-m_2 + n_2)y_2\,, \\
a_6 &= (m_2 - n_2)x_2 + (-m_2 - 2n_2)y_2\,, \\
b_1 &= (m_2 + 2n_2)x_2 + (-m_2 + n_2)y_2\,, \\
b_2 &= (-2m_2 - n_2)x_2 + (-m_2 - 2n_2)y_2\,, \\
b_3 &= (m_2 - n_2)x_2 + (2m_2 + n_2)y_2\,, \\
b_4 &= (m_1 - n_1)x_1 + (-m_1 - 2n_1)y_1\,, \\
b_5 &= (-2m_1 - n_1)x_1 + (-m_1 + n_1)y_1\,, \\
b_6 &= (m_1 + 2n_1)x_1 + (2m_1 + n_1)y_1\,,
\end{aligned} \tag{2.6}
$$

then

$$\sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r\,, \tag{2.7}$$

is identically satisfied for $r = 1, 2$ and 4. Therefore, to obtain a solution of (2.1), we only have to choose m_i, n_i, x_i, y_i, such that (2.7) also holds for $r = 3$ and, at the same time, the additional condition $a_6 = b_6$ is satisfied.

When $r = 3$, (2.7) reduces to the equation

$$m_1 n_1 (m_1 + n_1) x_1 y_1 (x_1 + y_1) = m_2 n_2 (m_2 + n_2) x_2 y_2 (x_2 + y_2) \tag{2.8}$$

which is to be solved together with the additional condition

$$(m_2 - n_2)x_2 + (-m_2 - 2n_2)y_2 = (m_1 + 2n_1)x_1 + (2m_1 + n_1)y_1\,. \tag{2.9}$$

To solve the simultaneous equations (2.8) and (2.9), we write

$$m_2 = tm_1\,, \quad n_2 = tn_1\,, \quad x_1 = px_2\,, \quad y_1 = qy_2\,, \tag{2.10}$$

when (2.8) is readily solved to get

$$x_2 = pq^2 - t^3\,, \quad y_2 = -p^2 q + t^3\,. \tag{2.11}$$

Next, we find x_1, y_1 from (2.10), then solve (2.9) for m_1, n_1 to get

$$m_1 = pq - 2pt + t^2\,, \quad n_1 = pq + pt - 2t^2\,, \tag{2.12}$$

and then (2.10) gives

$$m_2 = t(pq - 2pt + t^2)\,, \quad n_2 = t(pq + pt - 2t^2)\,. \tag{2.13}$$

We now substitute the values of $m_1, n_1, m_2, n_2, x_1, x_2, y_1, y_2$ in (2.6) to get the following non-symmetric solution of the Tarry-Escott problem of degree four:

$$(2.14)\quad \begin{aligned}
a_1 &= p^3q^3 - p^3q^2t - p^2qt^3 + pqt^4 + pt^5 - qt^5\,,\\
a_2 &= p^3q^2t - p^2q^2t^2 + p^2qt^3 - p^2t^4 - pq^2t^3 + qt^5\,,\\
a_3 &= -p^3q^3 + p^2q^2t^2 + p^2t^4 + pq^2t^3 - pqt^4 - pt^5\,,\\
a_4 &= -p^3q^2t + p^3qt^2 + p^2q^3t - pq^2t^3 - pt^5 + t^6\,,\\
a_5 &= -p^3qt^2 - p^2q^3t + p^2q^2t^2 + p^2qt^3 + pqt^4 - t^6\,,\\
b_1 &= -p^3qt^2 + p^2q^3t + p^2qt^3 - pq^2t^3 - pqt^4 + pt^5\,,\\
b_2 &= p^3q^2t - p^2q^3t + p^2q^2t^2 - p^2qt^3 - pt^5 + t^6\,,\\
b_3 &= -p^3q^2t + p^3qt^2 - p^2q^2t^2 + pq^2t^3 + pqt^4 - t^6\,,\\
b_4 &= p^3q^3 - p^3q^2t + p^2t^4 - pq^2t^3 - pt^5 + qt^5\,,\\
b_5 &= -p^3q^3 + p^2q^2t^2 + p^2qt^3 - p^2t^4 + pqt^4 - qt^5\,.
\end{aligned}$$

While this solution is in terms of polynomials of degree six in three parameters, it yields simpler solutions in terms of polynomials of degree three if we consider q and t as constants. For example, taking $q = 1$, $t = -1$, we get the following ideal non-symmetric solution of the Tarry-Escott problem of degree four:

$$(2.15)\quad \begin{aligned}
a_1 &= 2p^3 + p^2 + 1\,, & b_1 &= -p^3 - 2p^2 - p\,,\\
a_2 &= -p^3 - 3p^2 + p - 1\,, & b_2 &= -p^3 + 3p^2 + p + 1\,,\\
a_3 &= -p^3 + 2p^2 - p\,, & b_3 &= 2p^3 - p^2 - 1\,,\\
a_4 &= 2p^3 - p^2 + 2p + 1\,, & b_4 &= 2p^3 + p^2 + 2p - 1\,,\\
a_5 &= -p^3 + p^2 + p - 1\,, & b_5 &= -p^3 - p^2 + p + 1\,.
\end{aligned}$$

In this solution we may take p as a rational parameter. Integer solutions of (2.1) are obtained by multiplying any rational numerical solution by a suitable constant. Substituting $p = -2$ in the above solution, we get, after suitable re-arrangement, the following numerical solution:

$$(-23)^r + (-11)^r + (-7)^r + 9^r + 18^r = (-21)^r + (-17)^r + 2^r + 3^r + 19^r$$

where $r = 1, 2, 3, 4$. Adding the constant 24 to all the terms, we get the following solution in positive integers:

$$1^r + 13^r + 17^r + 33^r + 42^r = 3^r + 7^r + 26^r + 27^r + 43^r\,,$$

where $r = 1, 2, 3, 4$.

We may apply the theorem of Gloden [2, p. 24] to the three-parameter ideal non-symmetric solution obtained above to derive a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r , \qquad r = 1,2,3,4,6, \tag{2.16}$$

in terms of polynomials of degree six in three parameters. We, however, restrict ourselves to applying this theorem to the simpler solution (2.15), and obtain the following solution of the system of equations (2.16):

$$\begin{aligned} a_1 &= 9p^3 + 5p^2 - 3p + 5 , & b_1 &= -6p^3 - 10p^2 - 8p , \\ a_2 &= -6p^3 - 15p^2 + 2p - 5 , & b_2 &= -6p^3 + 15p^2 + 2p + 5 , \\ a_3 &= -6p^3 + 10p^2 - 8p , & b_3 &= 9p^3 - 5p^2 - 3p - 5 , \\ a_4 &= 9p^3 - 5p^2 + 7p + 5 , & b_4 &= 9p^3 + 5p^2 + 7p - 5 , \\ a_5 &= -6p^3 + 5p^2 + 2p - 5 , & b_5 &= -6p^3 - 5p^2 + 2p + 5 . \end{aligned} \tag{2.17}$$

When $p = -2$, this leads to the following solution of the system of equations (2.16):

$$(-101)^r + (-41)^r + (-21)^r + 59^r + 104^r = (-91)^r + (-71)^r + 24^r + 29^r + 109^r$$

where $r = 1,2,3,4,6$.

We note that additional parametric non-symmetric solutions of the Tarry-Escott problem of degree four may be obtained by taking a_i, b_i, as in (2.6), and instead of imposing the condition $a_6 = b_6$, we reduce one term on either side by solving (2.8) together with another condition such as $a_4 = b_6$ or $a_5 = b_6$. Solutions obtained in this manner are of degrees $6, 7$ or 8 in terms of three parameters.

3. IDEAL NON-SYMMETRIC SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FIVE

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree five, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r , \qquad r = 1,2,3,4,5 . \tag{3.1}$$