

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREES FOUR AND FIVE AND RELATED DIOPHANTINE SYSTEMS

Autor: Choudhry, Ajai

DOI: <https://doi.org/10.5169/seals-66681>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 02.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM
OF DEGREES FOUR AND FIVE
AND RELATED DIOPHANTINE SYSTEMS

by Ajai CHOUDHRY

ABSTRACT. In this paper, we obtain parametric ideal non-symmetric solutions in integers of the Tarry-Escott problem of degrees four and five, that is, of the system of simultaneous equations $\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r$, $r = 1, 2, \dots, k$ where k is 4 or 5. We use these non-symmetric solutions to obtain parametric solutions of the two diophantine systems $\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r$, $r = 1, 2, \dots, k, k+2$ where k is 4 or 5.

1. INTRODUCTION

This paper is a sequel to my earlier paper [1] regarding the Tarry-Escott problem. It would be recalled that very little is known about ideal non-symmetric solutions of the Tarry-Escott problem of degree k when $k > 3$. When $k = 4$, the only known parametric ideal non-symmetric solution of the Tarry-Escott problem is given in [1]. This solution is in terms of polynomials of degree 8 in two parameters. When $k = 5$, only a single numerical solution seems to have been published [2, p.27]. No non-symmetric solutions have been published for $k > 5$.

In this paper, we will obtain parametric ideal non-symmetric solutions of the Tarry-Escott problem of degrees four and five. The parametric solutions of the Tarry-Escott problem of degree four obtained in this paper are more general and much simpler as compared to the parametric solution of this problem given in [1].

It has already been shown in [1] how ideal non-symmetric solutions of the Tarry-Escott problem of degree k may be used to generate solutions of the system of equations

$$(1.1) \quad \sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \quad r = 1, 2, \dots, k, k+2$$

by applying a theorem of Gloden [2, p.24]. Applying this procedure to the non-symmetric ideal solutions of degrees four and five obtained in this paper, we get parametric solutions of (1.1) when $k = 4$ or $k = 5$.

2. IDEAL NON-SYMMETRIC SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree four, we have to obtain a solution of the system of equations

$$(2.1) \quad \sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r, \quad r = 1, 2, 3, 4.$$

We first observe that the system of equations

$$(2.2) \quad X_1^r + X_2^r + X_3^r = Y_1^r + Y_2^r + Y_3^r, \quad r = 1, 2, 4,$$

reduces to

$$(2.3) \quad X_1^2 + X_1X_2 + X_2^2 = Y_1^2 + Y_1Y_2 + Y_2^2,$$

if we take $X_3 = -X_1 - X_2$ and $Y_3 = -Y_1 - Y_2$. A solution of (2.3) in terms of arbitrary parameters m, n, x, y , is given by

$$(2.4) \quad \begin{aligned} X_1 &= (m + 2n)x + (-m + n)y, \\ X_2 &= (-2m - n)x + (-m - 2n)y, \\ Y_1 &= (m - n)x + (-m - 2n)y, \\ Y_2 &= (-2m - n)x + (-m + n)y, \end{aligned}$$

and we now get

$$(2.5) \quad \begin{aligned} X_3 &= (m - n)x + (2m + n)y, \\ Y_3 &= (m + 2n)x + (2m + n)y. \end{aligned}$$

It follows from this solution of the system of equations (2.2) that if we take

$$\begin{aligned}
(2.6) \quad & a_1 = (m_1 + 2n_1)x_1 + (-m_1 + n_1)y_1, \\
& a_2 = (-2m_1 - n_1)x_1 + (-m_1 - 2n_1)y_1, \\
& a_3 = (m_1 - n_1)x_1 + (2m_1 + n_1)y_1, \\
& a_4 = (m_2 + 2n_2)x_2 + (2m_2 + n_2)y_2, \\
& a_5 = (-2m_2 - n_2)x_2 + (-m_2 + n_2)y_2, \\
& a_6 = (m_2 - n_2)x_2 + (-m_2 - 2n_2)y_2, \\
& b_1 = (m_2 + 2n_2)x_2 + (-m_2 + n_2)y_2, \\
& b_2 = (-2m_2 - n_2)x_2 + (-m_2 - 2n_2)y_2, \\
& b_3 = (m_2 - n_2)x_2 + (2m_2 + n_2)y_2, \\
& b_4 = (m_1 - n_1)x_1 + (-m_1 - 2n_1)y_1, \\
& b_5 = (-2m_1 - n_1)x_1 + (-m_1 + n_1)y_1, \\
& b_6 = (m_1 + 2n_1)x_1 + (2m_1 + n_1)y_1,
\end{aligned}$$

then

$$(2.7) \quad \sum_{i=1}^6 a_i^r = \sum_{i=1}^6 b_i^r,$$

is identically satisfied for $r = 1, 2$ and 4 . Therefore, to obtain a solution of (2.1), we only have to choose m_i, n_i, x_i, y_i , such that (2.7) also holds for $r = 3$ and, at the same time, the additional condition $a_6 = b_6$ is satisfied.

When $r = 3$, (2.7) reduces to the equation

$$(2.8) \quad m_1 n_1 (m_1 + n_1) x_1 y_1 (x_1 + y_1) = m_2 n_2 (m_2 + n_2) x_2 y_2 (x_2 + y_2)$$

which is to be solved together with the additional condition

$$(2.9) \quad (m_2 - n_2)x_2 + (-m_2 - 2n_2)y_2 = (m_1 + 2n_1)x_1 + (2m_1 + n_1)y_1.$$

To solve the simultaneous equations (2.8) and (2.9), we write

$$(2.10) \quad m_2 = tm_1, \quad n_2 = tn_1, \quad x_1 = px_2, \quad y_1 = qy_2,$$

when (2.8) is readily solved to get

$$(2.11) \quad x_2 = pq^2 - t^3, \quad y_2 = -p^2q + t^3.$$

Next, we find x_1, y_1 from (2.10), then solve (2.9) for m_1, n_1 to get

$$(2.12) \quad m_1 = pq - 2pt + t^2, \quad n_1 = pq + pt - 2t^2,$$

and then (2.10) gives

$$(2.13) \quad m_2 = t(pq - 2pt + t^2), \quad n_2 = t(pq + pt - 2t^2).$$

We now substitute the values of $m_1, n_1, m_2, n_2, x_1, x_2, y_1, y_2$ in (2.6) to get the following non-symmetric solution of the Tarry-Escott problem of degree four:

$$\begin{aligned}
 (2.14) \quad a_1 &= p^3 q^3 - p^3 q^2 t - p^2 q t^3 + p q t^4 + p t^5 - q t^5, \\
 a_2 &= p^3 q^2 t - p^2 q^2 t^2 + p^2 q t^3 - p^2 t^4 - p q^2 t^3 + q t^5, \\
 a_3 &= -p^3 q^3 + p^2 q^2 t^2 + p^2 t^4 + p q^2 t^3 - p q t^4 - p t^5, \\
 a_4 &= -p^3 q^2 t + p^3 q t^2 + p^2 q^3 t - p q^2 t^3 - p t^5 + t^6, \\
 a_5 &= -p^3 q t^2 - p^2 q^3 t + p^2 q^2 t^2 + p^2 q t^3 + p q t^4 - t^6, \\
 b_1 &= -p^3 q t^2 + p^2 q^3 t + p^2 q t^3 - p q^2 t^3 - p q t^4 + p t^5, \\
 b_2 &= p^3 q^2 t - p^2 q^3 t + p^2 q^2 t^2 - p^2 q t^3 - p t^5 + t^6, \\
 b_3 &= -p^3 q^2 t + p^3 q t^2 - p^2 q^2 t^2 + p q^2 t^3 + p q t^4 - t^6, \\
 b_4 &= p^3 q^3 - p^3 q^2 t + p^2 t^4 - p q^2 t^3 - p t^5 + q t^5, \\
 b_5 &= -p^3 q^3 + p^2 q^2 t^2 + p^2 q t^3 - p^2 t^4 + p q t^4 - q t^5.
 \end{aligned}$$

While this solution is in terms of polynomials of degree six in three parameters, it yields simpler solutions in terms of polynomials of degree three if we consider q and t as constants. For example, taking $q = 1, t = -1$, we get the following ideal non-symmetric solution of the Tarry-Escott problem of degree four:

$$\begin{aligned}
 (2.15) \quad a_1 &= 2p^3 + p^2 + 1, & b_1 &= -p^3 - 2p^2 - p, \\
 a_2 &= -p^3 - 3p^2 + p - 1, & b_2 &= -p^3 + 3p^2 + p + 1, \\
 a_3 &= -p^3 + 2p^2 - p, & b_3 &= 2p^3 - p^2 - 1, \\
 a_4 &= 2p^3 - p^2 + 2p + 1, & b_4 &= 2p^3 + p^2 + 2p - 1, \\
 a_5 &= -p^3 + p^2 + p - 1, & b_5 &= -p^3 - p^2 + p + 1.
 \end{aligned}$$

In this solution we may take p as a rational parameter. Integer solutions of (2.1) are obtained by multiplying any rational numerical solution by a suitable constant. Substituting $p = -2$ in the above solution, we get, after suitable re-arrangement, the following numerical solution:

$$(-23)^r + (-11)^r + (-7)^r + 9^r + 18^r = (-21)^r + (-17)^r + 2^r + 3^r + 19^r$$

where $r = 1, 2, 3, 4$. Adding the constant 24 to all the terms, we get the following solution in positive integers:

$$1^r + 13^r + 17^r + 33^r + 42^r = 3^r + 7^r + 26^r + 27^r + 43^r,$$

where $r = 1, 2, 3, 4$.

We may apply the theorem of Gloden [2, p. 24] to the three-parameter ideal non-symmetric solution obtained above to derive a solution of the system of equations

$$(2.16) \quad \sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r, \quad r = 1, 2, 3, 4, 6,$$

in terms of polynomials of degree six in three parameters. We, however, restrict ourselves to applying this theorem to the simpler solution (2.15), and obtain the following solution of the system of equations (2.16):

$$(2.17) \quad \begin{aligned} a_1 &= 9p^3 + 5p^2 - 3p + 5, & b_1 &= -6p^3 - 10p^2 - 8p, \\ a_2 &= -6p^3 - 15p^2 + 2p - 5, & b_2 &= -6p^3 + 15p^2 + 2p + 5, \\ a_3 &= -6p^3 + 10p^2 - 8p, & b_3 &= 9p^3 - 5p^2 - 3p - 5, \\ a_4 &= 9p^3 - 5p^2 + 7p + 5, & b_4 &= 9p^3 + 5p^2 + 7p - 5, \\ a_5 &= -6p^3 + 5p^2 + 2p - 5, & b_5 &= -6p^3 - 5p^2 + 2p + 5. \end{aligned}$$

When $p = -2$, this leads to the following solution of the system of equations (2.16):

$$(-101)^r + (-41)^r + (-21)^r + 59^r + 104^r = (-91)^r + (-71)^r + 24^r + 29^r + 109^r$$

where $r = 1, 2, 3, 4, 6$.

We note that additional parametric non-symmetric solutions of the Tarry-Escott problem of degree four may be obtained by taking a_i, b_i , as in (2.6), and instead of imposing the condition $a_6 = b_6$, we reduce one term on either side by solving (2.8) together with another condition such as $a_4 = b_6$ or $a_5 = b_6$. Solutions obtained in this manner are of degrees 6, 7 or 8 in terms of three parameters.

3. IDEAL NON-SYMMETRIC SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FIVE

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree five, we have to obtain a solution of the system of equations

$$(3.1) \quad \sum_{i=1}^6 a_i^r = \sum_{i=1}^6 b_i^r, \quad r = 1, 2, 3, 4, 5.$$

We will choose $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ as in (2.6) when (3.1) holds identically for $r = 1, 2, 4$. For $r = 3$, equation (3.1) reduces to (2.8) while for $r = 5$ it reduces to the equation:

$$(3.2) \quad m_1 n_1 (m_1 + n_1) x_1 y_1 (x_1 + y_1) (m_1^2 + m_1 n_1 + n_1^2) (x_1^2 + x_1 y_1 + y_1^2) \\ = m_2 n_2 (m_2 + n_2) x_2 y_2 (x_2 + y_2) (m_2^2 + m_2 n_2 + n_2^2) (x_2^2 + x_2 y_2 + y_2^2).$$

It therefore suffices to solve equation (2.8) together with the following equation:

$$(3.3) \quad (m_1^2 + m_1 n_1 + n_1^2) (x_1^2 + x_1 y_1 + y_1^2) = (m_2^2 + m_2 n_2 + n_2^2) (x_2^2 + x_2 y_2 + y_2^2).$$

We take x_1, y_1, m_1, m_2, n_2 such that

$$(3.4) \quad \begin{aligned} x_1 &= (t^2 + t - 1)x_2, \\ y_1 &= (t + 1)^2 y_2, \\ m_1 &= tx_2 + ty_2, \\ m_2 &= (t^2 + t - 1)x_2 + (t + 1)^2 y_2, \\ n_2 &= (-t^2 - t)n_1. \end{aligned}$$

Substituting these values of x_1, y_1, m_1, m_2, n_2 in (2.8) and solving for n_1 , we get

$$(3.5) \quad n_1 = -\frac{(t^4 + 2t^3 + t^2 - 1)x_2 + (t^4 + 2t^3 + t^2 + t + 1)y_2}{t^3 + t^2 - t - 1},$$

and now (3.4) gives

$$(3.6) \quad n_2 = \frac{(t^5 + 2t^4 + t^3 - t)x_2 + (t^5 + 2t^4 + t^3 + t^2 + t)y_2}{t^2 - 1}.$$

On substituting the values of n_1, n_2 given by (3.5) and (3.6), and the values of x_1, y_1, m_1, m_2 given by (3.4) in equation (3.3), we get the equation:

$$(tx_2 + (t + 1)y_2)((t^2 + t - 1)x_2 + (t + 1)y_2)((t^2 + t - 1)x_2 + (t^2 + t)y_2) \\ \times (t + 2)((2t^5 + 5t^4 + 3t^3 - t^2 - t + 1)x_2 + (t^5 - 6t^3 - 8t^2 - 4t - 1)y_2) = 0.$$

Equating any of the first four factors on the left-hand side of this equation to zero leads either to trivial solutions or to known symmetric solutions of the Tarry-Escott problem of degree five. However, on equating the last factor to zero, we get

$$(3.7) \quad \begin{aligned} x_2 &= t^5 - 6t^3 - 8t^2 - 4t - 1, \\ y_2 &= -(2t^5 + 5t^4 + 3t^3 - t^2 - t + 1), \end{aligned}$$

and now, using the relations (3.4), (3.5) and (3.6), we get the values of $x_1, y_1, m_1, m_2, n_1, n_2$. The values of $x_1, y_1, x_2, y_2, m_1, m_2, n_1, n_2$, may now be substituted in (2.6) to get a non-symmetric solution of the Tarry-Escott problem of degree five. After removing the common factors, this solution may be written as follows:

$$\begin{aligned}
 a_1 &= -3t^{11} - 20t^{10} - 58t^9 - 94t^8 - 106t^7 - 100t^6 - 40t^5 \\
 &\quad + 50t^4 + 50t^3 + 2t^2 - 5t, \\
 a_2 &= 3t^{11} + 16t^{10} + 38t^9 + 71t^8 + 128t^7 + 149t^6 + 56t^5 \\
 &\quad - 37t^4 - 22t^3 + 5t^2 + t - 3, \\
 a_3 &= 4t^{10} + 20t^9 + 23t^8 - 22t^7 - 49t^6 - 16t^5 \\
 &\quad - 13t^4 - 28t^3 - 7t^2 + 4t + 3, \\
 a_4 &= 8t^{10} + 52t^9 + 127t^8 + 148t^7 + 85t^6 + 22t^5 \\
 &\quad + t^4 - 14t^3 - 23t^2 - 4t + 3, \\
 a_5 &= 3t^{11} + 14t^{10} + 16t^9 - 29t^8 - 98t^7 - 89t^6 + 4t^5 + 55t^4 \\
 &\quad + 40t^3 + 13t^2 - 7t - 3, \\
 a_6 &= -3t^{11} - 22t^{10} - 68t^9 - 98t^8 - 50t^7 + 4t^6 - 26t^5 \\
 &\quad - 56t^4 - 26t^3 + 10t^2 + 11t, \\
 b_1 &= 2t^{10} + 13t^9 + 58t^8 + 151t^7 + 190t^6 + 79t^5 \\
 &\quad - 44t^4 - 41t^3 - 2t^2 - t, \\
 b_2 &= -3t^{11} - 19t^{10} - 56t^9 - 116t^8 - 164t^7 - 104t^6 + 34t^5 \\
 &\quad + 76t^4 + 28t^3 + 4t^2 - t - 3, \\
 b_3 &= 3t^{11} + 17t^{10} + 43t^9 + 58t^8 + 13t^7 - 86t^6 - 113t^5 \\
 &\quad - 32t^4 + 13t^3 - 2t^2 + 2t + 3, \\
 b_4 &= 10t^{10} + 59t^9 + 140t^8 + 167t^7 + 98t^6 - t^5 \\
 &\quad - 58t^4 - 31t^3 + 14t^2 + 7t, \\
 b_5 &= -3t^{11} - 23t^{10} - 67t^9 - 88t^8 - 25t^7 + 68t^6 + 71t^5 \\
 &\quad + 14t^4 - 13t^3 - 16t^2 - 2t + 3, \\
 b_6 &= 3t^{11} + 13t^{10} + 8t^9 - 52t^8 - 142t^7 - 166t^6 - 70t^5 \\
 &\quad + 44t^4 + 44t^3 + 2t^2 - 5t - 3.
 \end{aligned}
 \tag{3.8}$$

Here t is an arbitrary rational parameter and integer solutions of (3.1) are obtained by multiplying any rational numerical solution by a suitable constant. The a_i, b_i obtained above satisfy the relation $\sum_{i=1}^6 a_i = \sum_{i=1}^6 b_i = 0$.

It therefore follows from the theorem of Gloden [2, p.24] that these a_i, b_i must also satisfy the relation

$$\sum_{i=1}^6 a_i^7 = \sum_{i=1}^6 b_i^7.$$

This is also verified by direct computation. Hence the a_i, b_i given by (3.8) constitute a solution of the following system of equations:

$$\sum_{i=1}^6 a_i^r = \sum_{i=1}^6 b_i^r, \quad r = 1, 2, 3, 4, 5, 7.$$

As a numerical example, when $t = -3$, we get, after removal of common factors and suitable re-arrangement, the following solution

$$\begin{aligned} (-19323)^r + (-18689)^r + 3117^r + 5111^r + 14212^r + 15572^r \\ = (-20023)^r + (-17828)^r + 1017^r + 9787^r + 10236^r + 16811^r \end{aligned}$$

where $r = 1, 2, 3, 4, 5, 7$.

REFERENCES

- [1] CHOUDHRY, A. Ideal Solutions of the Tarry-Escott problem of degree four and a related diophantine system. *L'Enseignement Math. (2)* 46 (2000), 313–323.
- [2] GLODEN, A. *Mehrgradige Gleichungen*. Noordhoff, Groningen, 1944.

(Reçu le 27 novembre 2002)

Ajai Choudhry

High Commissioner
 High Commission of India
 P.O. Box 439, Mail Processing Centre
 Airport Lama, Berakas
 Bandar Seri Begawan, BB3577
 Brunei
e-mail: ajaic203@yahoo.com