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We now turn to the L^2 -index of Section 2. It extends to a homomorphism

$$\text{Index}_G: K_0(BG) \rightarrow \mathbf{R}$$

as follows. Each $x \in K_0(BG)$ is of the form $f_*(y)$ for some $y = [D] \in K_0(M)$, $f: M \rightarrow BG$, M a closed smooth manifold and D an elliptic operator on M . Let \tilde{D} be the lifted operator to \tilde{M} , the G -covering space induced by $f: M \rightarrow BG$. Then put

$$\text{Index}_G(x) := \text{Index}_G(\tilde{D}).$$

One checks that $\text{Index}_G(x)$ is indeed well-defined, either by direct computation, or by identifying it with $\tau(x)$, where τ denotes the composite of the assembly map $K_0(BG) \rightarrow K_0(C_r^*G)$ with the natural trace $K_0(C_r^*G) \rightarrow \mathbf{R}$ (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

LEMMA 4.2. *For $H < G$ the following diagram commutes:*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}_H} & \mathbf{R} \\ \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}_G} & \mathbf{R}. \quad \square \end{array}$$

Atiyah's L^2 -Index Theorem 2.1 for a given G can now be expressed as the statement (as already observed in [10])

$$\text{Index}_G = \text{Index}: K_0(BG) \rightarrow \mathbf{R}.$$

5. ALGEBRAIC PROOF OF ATIYAH'S L^2 -INDEX THEOREM

Recall that a group A is said to be *acyclic* if $H_*(BA, \mathbf{Z}) = 0$ for $* > 0$. For G a countable group, there exists an embedding $G \rightarrow A_G$ into a countable acyclic group A_G . There are many constructions of such a group A_G available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension ΣBA_G is contractible, and therefore the inclusion $\{e\} \rightarrow A_G$

induces an isomorphism

$$K_0(B\{e\}) \xrightarrow{\cong} K_0(BA_G).$$

Our strategy is as follows. We show that the Atiyah L^2 -Index Theorem holds in the special case of acyclic groups, and finish the proof combining the above embedding of a group into an acyclic group.

Proof of Theorem 2.1. If a group A is acyclic, the equation $\text{Index}_A = \text{Index}$ follows from the diagram

$$\begin{array}{ccccc} K_0(BA) & \xrightarrow{\text{Index}_A} & \mathbf{R} & \xleftarrow{\text{Index}} & K_0(BA) \\ \cong \uparrow & & \uparrow & & \cong \uparrow \\ K_0(B\{e\}) & \xrightarrow[\cong]{\text{Index}_{\{e\}}} & \mathbf{Z} & \xleftarrow[\cong]{\text{Index}} & K_0(B\{e\}) \end{array}$$

because $\text{Index}_{\{e\}} = \text{Index}$ on the bottom line. For a general group G , consider an embedding into an acyclic group A_G and complete the proof by using Lemma 3.1, together with Lemmas 4.1 and 4.2.

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