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We now turn to the  $L^2$ -index of Section 2. It extends to a homomorphism

$$\text{Index}_G: K_0(BG) \rightarrow \mathbf{R}$$

as follows. Each  $x \in K_0(BG)$  is of the form  $f_*(y)$  for some  $y = [D] \in K_0(M)$ ,  $f: M \rightarrow BG$ ,  $M$  a closed smooth manifold and  $D$  an elliptic operator on  $M$ . Let  $\tilde{D}$  be the lifted operator to  $\tilde{M}$ , the  $G$ -covering space induced by  $f: M \rightarrow BG$ . Then put

$$\text{Index}_G(x) := \text{Index}_G(\tilde{D}).$$

One checks that  $\text{Index}_G(x)$  is indeed well-defined, either by direct computation, or by identifying it with  $\tau(x)$ , where  $\tau$  denotes the composite of the assembly map  $K_0(BG) \rightarrow K_0(C_r^*G)$  with the natural trace  $K_0(C_r^*G) \rightarrow \mathbf{R}$  (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

LEMMA 4.2. *For  $H < G$  the following diagram commutes:*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}_H} & \mathbf{R} \\ \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}_G} & \mathbf{R}. \quad \square \end{array}$$

Atiyah's  $L^2$ -Index Theorem 2.1 for a given  $G$  can now be expressed as the statement (as already observed in [10])

$$\text{Index}_G = \text{Index}: K_0(BG) \rightarrow \mathbf{R}.$$

## 5. ALGEBRAIC PROOF OF ATIYAH'S $L^2$ -INDEX THEOREM

Recall that a group  $A$  is said to be *acyclic* if  $H_*(BA, \mathbf{Z}) = 0$  for  $* > 0$ . For  $G$  a countable group, there exists an embedding  $G \rightarrow A_G$  into a countable acyclic group  $A_G$ . There are many constructions of such a group  $A_G$  available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension  $\Sigma BA_G$  is contractible, and therefore the inclusion  $\{e\} \rightarrow A_G$

induces an isomorphism

$$K_0(B\{e\}) \xrightarrow{\cong} K_0(BA_G).$$

Our strategy is as follows. We show that the Atiyah  $L^2$ -Index Theorem holds in the special case of acyclic groups, and finish the proof combining the above embedding of a group into an acyclic group.

*Proof of Theorem 2.1.* If a group  $A$  is acyclic, the equation  $\text{Index}_A = \text{Index}$  follows from the diagram

$$\begin{array}{ccccc} K_0(BA) & \xrightarrow{\text{Index}_A} & \mathbf{R} & \xleftarrow{\text{Index}} & K_0(BA) \\ \cong \uparrow & & \uparrow & & \cong \uparrow \\ K_0(B\{e\}) & \xrightarrow[\cong]{\text{Index}_{\{e\}}} & \mathbf{Z} & \xleftarrow[\cong]{\text{Index}} & K_0(B\{e\}) \end{array}$$

because  $\text{Index}_{\{e\}} = \text{Index}$  on the bottom line. For a general group  $G$ , consider an embedding into an acyclic group  $A_G$  and complete the proof by using Lemma 3.1, together with Lemmas 4.1 and 4.2.

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