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Autor: Chatterji, Indira / Mislin, Guido
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For R a fixed set of representatives for G/H , the map

$$\begin{aligned}\varphi_R: \text{Ind}_H^G(S_{\tilde{D}}) &\rightarrow S_{\tilde{D}} \\ f &\mapsto \{f(r)\}_{r \in R}\end{aligned}$$

is well-defined by H -equivariance of the elements of $S_{\tilde{D}}$ and one checks that it defines a G -equivariant isometric bijection. Similarly for the adjoint operators.

The following example is a particular case of the previous lemma.

EXAMPLE 3.2. Let us look at the case $\tilde{M} = M \times G$. A section $\tilde{s} \in C_c^\infty(\tilde{M}, \pi^*E)$ is an element $\tilde{s} = \{s_g\}_{g \in G}$ where $s_g \in C^\infty(M, E)$ and $s_g = 0$ for all but finitely many g 's. Note that $L^2(\tilde{M}, \pi^*E)$ can be identified with $\ell^2(G) \otimes L^2(M, E)$. Now

$$\tilde{D}\tilde{s} = \{Ds_g\}_{g \in G} \in C_c^\infty(\tilde{M}, \pi^*F)$$

and hence $S_{\tilde{D}}$ may be identified with $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$, where $d = \dim_{\mathbb{C}}(S_D)$. In this identification the projection P onto $S_{\tilde{D}}$ becomes the identity in $M_d(\mathcal{N}(G))$ and thus

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^d \langle e, e \rangle = d = \dim_{\mathbb{C}}(S_D).$$

A similar argument for D^* shows that in this case not only does the L^2 -Index of \tilde{D} coincide with the Index of D , but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.

4. ON K -HOMOLOGY

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator D on the closed manifold M can also be used to define an element $[D] \in K_0(M)$, the K -homology of M , and according to Baum and Douglas [4], all elements of $K_0(M)$ are of the form $[D]$. The index defined in Section 2 extends to a well-defined

homomorphism (cf. [4])

$$\text{Index}: K_0(M) \rightarrow \mathbf{Z},$$

such that $\text{Index}([D]) = \text{Index}(D)$. On the other hand, the projection $\text{pr}: M \rightarrow \{pt\}$ induces, after identifying $K_0(\{pt\})$ with \mathbf{Z} , a homomorphism

$$(*) \quad \text{pr}_*: K_0(M) \rightarrow \mathbf{Z},$$

which, as explained in [4], satisfies

$$\text{pr}_*([D]) = \text{Index}([D]).$$

More generally (cf. [4]), for a not necessarily finite CW-complex X , every $x \in K_0(X)$ is of the form $f_*[D]$ for some $f: M \rightarrow X$, and $K_0(X)$ is obtained as a colimit over $K_0(M_\alpha)$, where the M_α form a directed system consisting of closed Riemannian manifolds (these homology groups $K_0(X)$ are naturally isomorphic to the ones defined using the Bott spectrum; sometimes, they are referred to as K -homology groups with *compact supports*). The index map from above extends to a homomorphism

$$\text{Index}: K_0(X) \rightarrow \mathbf{Z},$$

such that $\text{Index}(x) = \text{Index}([D])$ if $x = f_*[D]$, with $f: M \rightarrow X$.

We now consider the case of $X = BG$, the classifying space of the discrete group G , and obtain thus for any $f: M \rightarrow BG$ a commutative diagram

$$\begin{array}{ccc} K_0(M) & \xrightarrow{\text{Index}} & \mathbf{Z} \\ f_* \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbf{Z}. \end{array}$$

Note that (*) from above implies the following naturality property for the index homomorphism.

LEMMA 4.1. *For any homomorphism $\varphi: H \rightarrow G$ one has a commutative diagram*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}} & \mathbf{Z} \\ (B\varphi)_* \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbf{Z}. \quad \square \end{array}$$

We now turn to the L^2 -index of Section 2. It extends to a homomorphism

$$\text{Index}_G: K_0(BG) \rightarrow \mathbf{R}$$

as follows. Each $x \in K_0(BG)$ is of the form $f_*(y)$ for some $y = [D] \in K_0(M)$, $f: M \rightarrow BG$, M a closed smooth manifold and D an elliptic operator on M . Let \tilde{D} be the lifted operator to \tilde{M} , the G -covering space induced by $f: M \rightarrow BG$. Then put

$$\text{Index}_G(x) := \text{Index}_G(\tilde{D}).$$

One checks that $\text{Index}_G(x)$ is indeed well-defined, either by direct computation, or by identifying it with $\tau(x)$, where τ denotes the composite of the assembly map $K_0(BG) \rightarrow K_0(C_r^*G)$ with the natural trace $K_0(C_r^*G) \rightarrow \mathbf{R}$ (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

LEMMA 4.2. *For $H < G$ the following diagram commutes:*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}_H} & \mathbf{R} \\ \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}_G} & \mathbf{R}. \quad \square \end{array}$$

Atiyah's L^2 -Index Theorem 2.1 for a given G can now be expressed as the statement (as already observed in [10])

$$\text{Index}_G = \text{Index}: K_0(BG) \rightarrow \mathbf{R}.$$

5. ALGEBRAIC PROOF OF ATIYAH'S L^2 -INDEX THEOREM

Recall that a group A is said to be *acyclic* if $H_*(BA, \mathbf{Z}) = 0$ for $* > 0$. For G a countable group, there exists an embedding $G \rightarrow A_G$ into a countable acyclic group A_G . There are many constructions of such a group A_G available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension ΣBA_G is contractible, and therefore the inclusion $\{e\} \rightarrow A_G$