

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ATIYAH'S L^2 -INDEX THEOREM
Autor: Chatterji, Indira / Mislin, Guido
Kapitel: 2. REVIEW OF THE L^2 -INDEX THEOREM
DOI: <https://doi.org/10.5169/seals-66679>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 01.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ATIYAH'S L^2 -INDEX THEOREM

by Indira CHATTERJI and Guido MISLIN

1. INTRODUCTION

The L^2 -Index Theorem of Atiyah [1] expresses the index of an elliptic operator on a closed manifold M in terms of the G -equivariant index of some regular covering \tilde{M} of M , with G the group of covering transformations. Atiyah's proof is analytic in nature. Our proof is algebraic and involves an embedding of a given group into an acyclic one, together with naturality properties of the indices.

2. REVIEW OF THE L^2 -INDEX THEOREM

The main reference for this section is Atiyah's paper [1]. All manifolds considered are smooth Riemannian, without boundary. Covering spaces of manifolds carry the induced smooth and Riemannian structure. Let M be a closed manifold and let E, F denote two complex (Hermitian) vector bundles over M . Consider an elliptic pseudo-differential operator

$$D: C^\infty(M, E) \rightarrow C^\infty(M, F)$$

acting on the smooth sections of the vector bundles. One defines its space of solutions

$$S_D = \{s \in C^\infty(M, E) \mid Ds = 0\} .$$

The complex vector space S_D has finite dimension (see [13]), and so has S_{D^*} the space of solutions of the adjoint D^* of D where

$$D^*: C^\infty(M, F) \rightarrow C^\infty(M, E)$$

is the unique continuous linear map satisfying

$$\langle Ds, s' \rangle = \int_M \langle Ds(m), s'(m) \rangle_F dm = \langle s, D^* s' \rangle = \int_M \langle s(m), D^* s'(m) \rangle_E dm$$

for all $s \in C^\infty(M, E)$, $s' \in C^\infty(M, F)$. One now defines the *index* of D as follows:

$$\text{Index}(D) = \dim_{\mathbf{C}}(S_D) - \dim_{\mathbf{C}}(S_{D^*}) \in \mathbf{Z}.$$

An explicit formula for $\text{Index}(D)$ is given by the famous Atiyah-Singer Theorem (cf. [2]). Consider a not necessarily connected, regular covering $\pi: \tilde{M} \rightarrow M$ with countable covering transformation group G . The projection π can be used to define an elliptic operator

$$\tilde{D} := \pi^*(D): C_c^\infty(\tilde{M}, \pi^*E) \rightarrow C_c^\infty(\tilde{M}, \pi^*F).$$

Denote by $S_{\tilde{D}}$ the closure of $\{s \in C_c^\infty(\tilde{M}, \pi^*E) \mid \tilde{D}s = 0\}$ in $L^2(\tilde{M}, \pi^*E)$. Let \tilde{D}^* denote the adjoint of \tilde{D} . The space $S_{\tilde{D}}$ is not necessarily finite dimensional, but being a closed G -invariant subspace of the L^2 -completion $L^2(\tilde{M}, \pi^*E)$ of the space of smooth sections with compact supports $C_c^\infty(\tilde{M}, \pi^*E)$, its von Neumann dimension is therefore defined as follows. Write

$$\mathcal{N}(G) = \{P: \ell^2(G) \rightarrow \ell^2(G) \text{ bounded and } G\text{-invariant}\}$$

for the group von Neumann algebra of G , where G acts on $\ell^2(G)$ via the right regular representation. Then $S_{\tilde{D}}$ is a finitely generated Hilbert G -module and hence can be represented by an idempotent matrix $P = (p_{ij}) \in M_n(\mathcal{N}(G))$ (recall that a finitely generated Hilbert G -module is isometrically G -isomorphic to a Hilbert G -subspace of the Hilbert space $\ell^2(G)^n$ for some $n \geq 1$, see [9]). One then sets

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^n \langle p_{ii}(e), e \rangle = \kappa(P) \in \mathbf{R},$$

where by abuse of notation e denotes the element in $\ell^2(G)$ taking value 1 on the neutral element $e \in G$ and 0 elsewhere (see Eckmann's survey [9] on L^2 -cohomology for more on von Neumann dimensions). The map $\kappa: M_n(\mathcal{N}(G)) \rightarrow \mathbf{C}$ is the Kaplansky trace. One defines the L^2 -index of \tilde{D} by

$$\text{Index}_G(\tilde{D}) = \dim_G(S_{\tilde{D}}) - \dim_G(S_{\tilde{D}^*}).$$

We can now state Atiyah's L^2 -Index Theorem.

THEOREM 2.1 (Atiyah [1]). *For D an elliptic pseudo-differential operator on a closed Riemannian manifold M*

$$\text{Index}(D) = \text{Index}_G(\tilde{D})$$

for any countable group G and any lift \tilde{D} of D to a regular G -cover \tilde{M} of M .

In particular, the L^2 -index of \tilde{D} is always an integer, even though it is a priori given in terms of real numbers. The following serves as an illustration of the L^2 -Index Theorem.

EXAMPLE 2.2 (Atiyah's formula [1]). Let Ω^\bullet be the de Rham complex of complex valued differential forms on the closed connected manifold M and consider the de Rham differential $D = d + d^* : \Omega^{ev} \rightarrow \Omega^{odd}$. Let $\pi : \tilde{M} \rightarrow M$ be the universal cover of M so that $G = \pi_1(M)$. Then

- $\text{Index}(D) = \chi(M)$, the ordinary Euler characteristic of M .
- $\text{Index}_G(\tilde{D}) = \sum_j (-1)^j \beta^j(M)$, the L^2 -Euler characteristic of M .

The $\beta^j(M)$'s denote the L^2 -Betti numbers of M . Thus the L^2 -Index Theorem translates into Atiyah's formula

$$\chi(M) = \sum_j (-1)^j \beta^j(M).$$

We recall that the L^2 -Betti numbers $\beta^j(M)$ are in general not integers. For instance, if $\pi_1(M)$ is a finite group, one checks that

$$\beta^j(M) = \frac{1}{|\pi_1(M)|} b^j(\tilde{M}),$$

where $b^j(\tilde{M})$ stands for the ordinary j 'th Betti number of the universal cover \tilde{M} of M . In particular, for $1 < |\pi_1(M)| < \infty$, $\beta^0(M) = 1/|\pi_1(M)|$ is not an integer and the L^2 -Index Theorem reduces to the well-known fact that

$$\chi(M) = \frac{\chi(\tilde{M})}{|\pi_1(M)|}.$$

It is a conjecture (Atiyah Conjecture) that for a general closed connected manifold M the L^2 -Betti numbers $\beta^j(M)$ are always rational numbers, and even integers in case that $\pi_1(M)$ is torsion-free. For some interesting examples, which might lead to counterexamples, see Dicks and Schick [8].