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## ATIYAH'S $L^2$ -INDEX THEOREM

by Indira CHATTERJI and Guido MISLIN

### 1. INTRODUCTION

The  $L^2$ -Index Theorem of Atiyah [1] expresses the index of an elliptic operator on a closed manifold  $M$  in terms of the  $G$ -equivariant index of some regular covering  $\tilde{M}$  of  $M$ , with  $G$  the group of covering transformations. Atiyah's proof is analytic in nature. Our proof is algebraic and involves an embedding of a given group into an acyclic one, together with naturality properties of the indices.

### 2. REVIEW OF THE $L^2$ -INDEX THEOREM

The main reference for this section is Atiyah's paper [1]. All manifolds considered are smooth Riemannian, without boundary. Covering spaces of manifolds carry the induced smooth and Riemannian structure. Let  $M$  be a closed manifold and let  $E, F$  denote two complex (Hermitian) vector bundles over  $M$ . Consider an elliptic pseudo-differential operator

$$D: C^\infty(M, E) \rightarrow C^\infty(M, F)$$

acting on the smooth sections of the vector bundles. One defines its space of solutions

$$S_D = \{s \in C^\infty(M, E) \mid Ds = 0\}.$$

The complex vector space  $S_D$  has finite dimension (see [13]), and so has  $S_{D^*}$  the space of solutions of the adjoint  $D^*$  of  $D$  where

$$D^*: C^\infty(M, F) \rightarrow C^\infty(M, E)$$

is the unique continuous linear map satisfying

$$\langle Ds, s' \rangle = \int_M \langle Ds(m), s'(m) \rangle_F dm = \langle s, D^* s' \rangle = \int_M \langle s(m), D^* s'(m) \rangle_E dm$$

for all  $s \in C^\infty(M, E)$ ,  $s' \in C^\infty(M, F)$ . One now defines the *index* of  $D$  as follows:

$$\text{Index}(D) = \dim_{\mathbf{C}}(S_D) - \dim_{\mathbf{C}}(S_{D^*}) \in \mathbf{Z}.$$

An explicit formula for  $\text{Index}(D)$  is given by the famous Atiyah-Singer Theorem (cf. [2]). Consider a not necessarily connected, regular covering  $\pi: \tilde{M} \rightarrow M$  with countable covering transformation group  $G$ . The projection  $\pi$  can be used to define an elliptic operator

$$\tilde{D} := \pi^*(D): C_c^\infty(\tilde{M}, \pi^*E) \rightarrow C_c^\infty(\tilde{M}, \pi^*F).$$

Denote by  $S_{\tilde{D}}$  the closure of  $\{s \in C_c^\infty(\tilde{M}, \pi^*E) \mid \tilde{D}s = 0\}$  in  $L^2(\tilde{M}, \pi^*E)$ . Let  $\tilde{D}^*$  denote the adjoint of  $\tilde{D}$ . The space  $S_{\tilde{D}}$  is not necessarily finite dimensional, but being a closed  $G$ -invariant subspace of the  $L^2$ -completion  $L^2(\tilde{M}, \pi^*E)$  of the space of smooth sections with compact supports  $C_c^\infty(\tilde{M}, \pi^*E)$ , its von Neumann dimension is therefore defined as follows. Write

$$\mathcal{N}(G) = \{P: \ell^2(G) \rightarrow \ell^2(G) \text{ bounded and } G\text{-invariant}\}$$

for the group von Neumann algebra of  $G$ , where  $G$  acts on  $\ell^2(G)$  via the right regular representation. Then  $S_{\tilde{D}}$  is a finitely generated Hilbert  $G$ -module and hence can be represented by an idempotent matrix  $P = (p_{ij}) \in M_n(\mathcal{N}(G))$  (recall that a finitely generated Hilbert  $G$ -module is isometrically  $G$ -isomorphic to a Hilbert  $G$ -subspace of the Hilbert space  $\ell^2(G)^n$  for some  $n \geq 1$ , see [9]). One then sets

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^n \langle p_{ii}(e), e \rangle = \kappa(P) \in \mathbf{R},$$

where by abuse of notation  $e$  denotes the element in  $\ell^2(G)$  taking value 1 on the neutral element  $e \in G$  and 0 elsewhere (see Eckmann's survey [9] on  $L^2$ -cohomology for more on von Neumann dimensions). The map  $\kappa: M_n(\mathcal{N}(G)) \rightarrow \mathbf{C}$  is the Kaplansky trace. One defines the  $L^2$ -index of  $\tilde{D}$  by

$$\text{Index}_G(\tilde{D}) = \dim_G(S_{\tilde{D}}) - \dim_G(S_{\tilde{D}^*}).$$

We can now state Atiyah's  $L^2$ -Index Theorem.

THEOREM 2.1 (Atiyah [1]). For  $D$  an elliptic pseudo-differential operator on a closed Riemannian manifold  $M$

$$\text{Index}(D) = \text{Index}_G(\tilde{D})$$

for any countable group  $G$  and any lift  $\tilde{D}$  of  $D$  to a regular  $G$ -cover  $\tilde{M}$  of  $M$ .

In particular, the  $L^2$ -index of  $\tilde{D}$  is always an integer, even though it is a priori given in terms of real numbers. The following serves as an illustration of the  $L^2$ -Index Theorem.

EXAMPLE 2.2 (Atiyah's formula [1]). Let  $\Omega^\bullet$  be the de Rham complex of complex valued differential forms on the closed connected manifold  $M$  and consider the de Rham differential  $D = d + d^*: \Omega^{ev} \rightarrow \Omega^{odd}$ . Let  $\pi: \tilde{M} \rightarrow M$  be the universal cover of  $M$  so that  $G = \pi_1(M)$ . Then

- $\text{Index}(D) = \chi(M)$ , the ordinary Euler characteristic of  $M$ .
- $\text{Index}_G(\tilde{D}) = \sum_j (-1)^j \beta^j(M)$ , the  $L^2$ -Euler characteristic of  $M$ .

The  $\beta^j(M)$ 's denote the  $L^2$ -Betti numbers of  $M$ . Thus the  $L^2$ -Index Theorem translates into Atiyah's formula

$$\chi(M) = \sum_j (-1)^j \beta^j(M).$$

We recall that the  $L^2$ -Betti numbers  $\beta^j(M)$  are in general not integers. For instance, if  $\pi_1(M)$  is a finite group, one checks that

$$\beta^j(M) = \frac{1}{|\pi_1(M)|} b^j(\tilde{M}),$$

where  $b^j(\tilde{M})$  stands for the ordinary  $j$ 'th Betti number of the universal cover  $\tilde{M}$  of  $M$ . In particular, for  $1 < |\pi_1(M)| < \infty$ ,  $\beta^0(M) = 1/|\pi_1(M)|$  is not an integer and the  $L^2$ -Index Theorem reduces to the well-known fact that

$$\chi(M) = \frac{\chi(\tilde{M})}{|\pi_1(M)|}.$$

It is a conjecture (Atiyah Conjecture) that for a general closed connected manifold  $M$  the  $L^2$ -Betti numbers  $\beta^j(M)$  are always rational numbers, and even integers in case that  $\pi_1(M)$  is torsion-free. For some interesting examples, which might lead to counterexamples, see Dicks and Schick [8].



## 3. HILBERT MODULES

Recall that for  $H < G$  and  $X$  an  $H$ -space, the *induced*  $G$ -space is

$$G \times_H X = (G \times X)/H$$

where  $H$  acts on  $G \times X$  via  $h \cdot (g, x) = (gh^{-1}, hx)$  and the left  $G$ -action on  $G \times_H X$  is given by  $g \cdot [k, x] = [gk, x]$  (where  $[k, x]$  denotes the class of the pair  $(k, x) \in G \times X$  in  $G \times_H X$ ). For  $A \subseteq \ell^2(H)^n$  a Hilbert  $H$ -module one defines  $\text{Ind}_H^G(A)$ , the *induced* Hilbert  $G$ -module, as follows:

$$\text{Ind}_H^G(A) = \left\{ f: G \rightarrow A, \quad f(gh) = h^{-1}f(g), \quad \sum_{\gamma \in G/H} \|f(\gamma)\|^2 < \infty \right\}.$$

On  $\text{Ind}_H^G(A)$  the action of  $G$  is given as follows:

$$(\gamma \cdot f)(\mu) = f(\gamma^{-1}\mu), \quad \gamma, \mu \in G \text{ and } f \in \text{Ind}_H^G(A).$$

For  $\tilde{M}$  an  $H$ -free, cocompact Riemannian manifold and  $\tilde{D}$  an  $H$ -equivariant pseudo-differential operator on  $\tilde{M}$ , one can express the lift  $\bar{D}$  of  $\tilde{D}$  to  $\bar{M} = G \times_H \tilde{M}$  as follows. Fix a set  $R$  of representatives for  $G/H$  and write  $\pi: \bar{M} \rightarrow \tilde{M}$  for the projection; a section  $\bar{s} \in C_c^\infty(\bar{M}, \pi^*E)$  is a collection

$$\bar{s} = \{\tilde{s}_r\}_{r \in R},$$

where  $\tilde{s}_r \in C_c^\infty(\tilde{M}, E)$  is the zero section for all but finitely many  $r$ 's, and  $\bar{s}([g, \tilde{m}]) = \tilde{s}_r(h\tilde{m})$ , if  $[r, h\tilde{m}] = [g, \tilde{m}] \in G \times_H \tilde{M}$ . Now the lift  $\bar{D}$  of  $\tilde{D}$  to  $\bar{M} = G \times_H \tilde{M}$  satisfies

$$\bar{D}\bar{s} = \left\{ \tilde{D}\tilde{s}_r \right\}_{r \in R}.$$

LEMMA 3.1. *Let  $M$  be a closed Riemannian manifold,  $D$  a pseudo-differential operator on  $M$  and  $\tilde{M}$  a regular cover of  $M$  with countable transformation group  $H$ . Consider an inclusion  $H < G$  and form the regular cover  $\bar{M} = G \times_H \tilde{M}$  of  $M$ . Then for the lifts  $\tilde{D}$  of  $D$  to  $\tilde{M}$  and  $\bar{D}$  of  $\tilde{D}$  to  $\bar{M}$ ,*

$$\text{Index}_H(\tilde{D}) = \text{Index}_G(\bar{D}).$$

*Proof.* It is enough to see that  $S_{\bar{D}} \cong \text{Ind}_H^G(S_{\tilde{D}})$ . Indeed, it is well-known (see [9]) that for a Hilbert  $H$ -module  $A$  one has

$$\dim_H(A) = \dim_G(\text{Ind}_H^G(A)).$$

For  $R$  a fixed set of representatives for  $G/H$ , the map

$$\begin{aligned}\varphi_R: \text{Ind}_H^G(S_{\tilde{D}}) &\rightarrow S_{\tilde{D}} \\ f &\mapsto \{f(r)\}_{r \in R}\end{aligned}$$

is well-defined by  $H$ -equivariance of the elements of  $S_{\tilde{D}}$  and one checks that it defines a  $G$ -equivariant isometric bijection. Similarly for the adjoint operators.

The following example is a particular case of the previous lemma.

EXAMPLE 3.2. Let us look at the case  $\tilde{M} = M \times G$ . A section  $\tilde{s} \in C_c^\infty(\tilde{M}, \pi^*E)$  is an element  $\tilde{s} = \{s_g\}_{g \in G}$  where  $s_g \in C^\infty(M, E)$  and  $s_g = 0$  for all but finitely many  $g$ 's. Note that  $L^2(\tilde{M}, \pi^*E)$  can be identified with  $\ell^2(G) \otimes L^2(M, E)$ . Now

$$\tilde{D}\tilde{s} = \{Ds_g\}_{g \in G} \in C_c^\infty(\tilde{M}, \pi^*F)$$

and hence  $S_{\tilde{D}}$  may be identified with  $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$ , where  $d = \dim_{\mathbb{C}}(S_D)$ . In this identification the projection  $P$  onto  $S_{\tilde{D}}$  becomes the identity in  $M_d(\mathcal{N}(G))$  and thus

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^d \langle e, e \rangle = d = \dim_{\mathbb{C}}(S_D).$$

A similar argument for  $D^*$  shows that in this case not only does the  $L^2$ -Index of  $\tilde{D}$  coincide with the Index of  $D$ , but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.

#### 4. ON $K$ -HOMOLOGY

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator  $D$  on the closed manifold  $M$  can also be used to define an element  $[D] \in K_0(M)$ , the  $K$ -homology of  $M$ , and according to Baum and Douglas [4], all elements of  $K_0(M)$  are of the form  $[D]$ . The index defined in Section 2 extends to a well-defined

homomorphism (cf. [4])

$$\text{Index}: K_0(M) \rightarrow \mathbf{Z},$$

such that  $\text{Index}([D]) = \text{Index}(D)$ . On the other hand, the projection  $\text{pr}: M \rightarrow \{pt\}$  induces, after identifying  $K_0(\{pt\})$  with  $\mathbf{Z}$ , a homomorphism

$$(*) \quad \text{pr}_*: K_0(M) \rightarrow \mathbf{Z},$$

which, as explained in [4], satisfies

$$\text{pr}_*([D]) = \text{Index}([D]).$$

More generally (cf. [4]), for a not necessarily finite CW-complex  $X$ , every  $x \in K_0(X)$  is of the form  $f_*[D]$  for some  $f: M \rightarrow X$ , and  $K_0(X)$  is obtained as a colimit over  $K_0(M_\alpha)$ , where the  $M_\alpha$  form a directed system consisting of closed Riemannian manifolds (these homology groups  $K_0(X)$  are naturally isomorphic to the ones defined using the Bott spectrum; sometimes, they are referred to as  $K$ -homology groups with *compact supports*). The index map from above extends to a homomorphism

$$\text{Index}: K_0(X) \rightarrow \mathbf{Z},$$

such that  $\text{Index}(x) = \text{Index}([D])$  if  $x = f_*[D]$ , with  $f: M \rightarrow X$ .

We now consider the case of  $X = BG$ , the classifying space of the discrete group  $G$ , and obtain thus for any  $f: M \rightarrow BG$  a commutative diagram

$$\begin{array}{ccc} K_0(M) & \xrightarrow{\text{Index}} & \mathbf{Z} \\ f_* \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbf{Z}. \end{array}$$

Note that (\*) from above implies the following naturality property for the index homomorphism.

LEMMA 4.1. *For any homomorphism  $\varphi: H \rightarrow G$  one has a commutative diagram*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}} & \mathbf{Z} \\ (B\varphi)_* \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbf{Z}. \quad \square \end{array}$$

We now turn to the  $L^2$ -index of Section 2. It extends to a homomorphism

$$\text{Index}_G: K_0(BG) \rightarrow \mathbf{R}$$

as follows. Each  $x \in K_0(BG)$  is of the form  $f_*(y)$  for some  $y = [D] \in K_0(M)$ ,  $f: M \rightarrow BG$ ,  $M$  a closed smooth manifold and  $D$  an elliptic operator on  $M$ . Let  $\tilde{D}$  be the lifted operator to  $\tilde{M}$ , the  $G$ -covering space induced by  $f: M \rightarrow BG$ . Then put

$$\text{Index}_G(x) := \text{Index}_G(\tilde{D}).$$

One checks that  $\text{Index}_G(x)$  is indeed well-defined, either by direct computation, or by identifying it with  $\tau(x)$ , where  $\tau$  denotes the composite of the assembly map  $K_0(BG) \rightarrow K_0(C_r^*G)$  with the natural trace  $K_0(C_r^*G) \rightarrow \mathbf{R}$  (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

LEMMA 4.2. *For  $H < G$  the following diagram commutes:*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}_H} & \mathbf{R} \\ \downarrow & & \parallel \\ K_0(BG) & \xrightarrow{\text{Index}_G} & \mathbf{R}. \quad \square \end{array}$$

Atiyah's  $L^2$ -Index Theorem 2.1 for a given  $G$  can now be expressed as the statement (as already observed in [10])

$$\text{Index}_G = \text{Index}: K_0(BG) \rightarrow \mathbf{R}.$$

## 5. ALGEBRAIC PROOF OF ATIYAH'S $L^2$ -INDEX THEOREM

Recall that a group  $A$  is said to be *acyclic* if  $H_*(BA, \mathbf{Z}) = 0$  for  $* > 0$ . For  $G$  a countable group, there exists an embedding  $G \rightarrow A_G$  into a countable acyclic group  $A_G$ . There are many constructions of such a group  $A_G$  available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension  $\Sigma BA_G$  is contractible, and therefore the inclusion  $\{e\} \rightarrow A_G$

induces an isomorphism

$$K_0(B\{e\}) \xrightarrow{\cong} K_0(BA_G).$$

Our strategy is as follows. We show that the Atiyah  $L^2$ -Index Theorem holds in the special case of acyclic groups, and finish the proof combining the above embedding of a group into an acyclic group.

*Proof of Theorem 2.1.* If a group  $A$  is acyclic, the equation  $\text{Index}_A = \text{Index}$  follows from the diagram

$$\begin{array}{ccccc} K_0(BA) & \xrightarrow{\text{Index}_A} & \mathbf{R} & \xleftarrow{\text{Index}} & K_0(BA) \\ \cong \uparrow & & \uparrow & & \cong \uparrow \\ K_0(B\{e\}) & \xrightarrow[\cong]{\text{Index}_{\{e\}}} & \mathbf{Z} & \xleftarrow[\cong]{\text{Index}} & K_0(B\{e\}) \end{array}$$

because  $\text{Index}_{\{e\}} = \text{Index}$  on the bottom line. For a general group  $G$ , consider an embedding into an acyclic group  $A_G$  and complete the proof by using Lemma 3.1, together with Lemmas 4.1 and 4.2.

## REFERENCES

- [1] ATIYAH, M.F. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque* 32–3 (1976), 43–72.
- [2] ATIYAH, M.F. and I.M. SINGER. The index of elliptic operators III. *Ann. of Math.* (2) 87 (1968), 546–604.
- [3] BAUM, P. and A. CONNES.  $K$ -theory for Lie groups and foliations. *L'Enseignement Math.* (2) 46 (2000), 3–42.
- [4] BAUM, P. and R. DOUGLAS.  $K$ -homology and index theory. *Proceedings of Symposia in Pure Mathematics* 38, Part 1 (1982), 117–173.
- [5] BERRICK, A.J. and K. VARADARAJAN. Binate towers of groups. *Arch. Math.* 62 (1994), 97–111.
- [6] BERRICK, A.J., I. CHATTERJI and G. MISLIN. From acyclic groups to the Bass Conjecture for amenable groups. (Submitted for publication 2002.)
- [7] BERRICK, A.J. The acyclic group dichotomy. (Preprint in preparation.)
- [8] DICKS, W. and T. SCHICK. The spectral measure of certain elements of the complex group ring of a wreath product. *Geom. Dedicata* 93 (2002), 121–137.
- [9] ECKMANN, B. Introduction to  $l_2$ -methods in topology: reduced  $l_2$ -homology, harmonic chains,  $l_2$ -Betti numbers. (Notes prepared by Guido Mislin.) *Israel J. Math.* 117 (2000), 183–219.
- [10] HIGSON, N. and J. ROE. *Analytic  $K$ -Homology*. Oxford Mathematical Monographs, Oxford University Press, 2000.

- [11] KAN, D.M. and W.P. THURSTON. Every connected space has the homology of a  $K(\pi, 1)$ . *Topology* 15 (1976), 253–258.
- [12] KASPAROV, G.  $K$ -theory, group  $C^*$ -algebras, and higher signatures (Conspetus). Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 101–146. London Math. Soc. Lecture Note Ser. 226. Cambridge Univ. Press, 1995.
- [13] SOLOVYOV, Y.P. and E.V. TROITSKY.  *$C^*$ -Algebras and Elliptic Operators in Differential Topology*. (Translated from the 1996 Russian original by Troitsky.) Translations of Mathematical Monographs, 192. Amer. Math.Soc., Providence (R.I.), 2001.
- [14] VALETTE, A. *Introduction to the Baum-Connes Conjecture*. (Notes taken by Indira Chatterji. With an appendix by Guido Mislin.) Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002.

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