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## 5. SPLITTING OF THE EXTENSION ASSOCIATED TO A NONCONNECTED COMPACT LIE GROUP

Let  $G$  still denote a compact Lie group with a nonabelian connected component; we also assume the other notations introduced previously. In this final section, we make a few observations on the following problem: when is the natural extension associated to  $G$  split, i.e. when is  $G$  isomorphic to a semidirect product  $G_0 \rtimes \Gamma$ ? Our aim is to relate this problem to the rest of this work. For a deeper analysis one should consult Chapter 6 in the book by Hofmann and Morris [14]. We start with a structure theorem for compact Lie groups based on centralizers of principal subgroups, similar to the “Sandwich Theorem for compact Lie groups” (see [14], Corollary 6.75, p.272). This theorem shows that any compact Lie group is “sandwiched” in between two semidirect products closely related to it. We then recall a theorem of de Siebenthal and compare, in some particular cases, the question of the splitting of the extension associated to  $G$  to that of the extensions associated to the normalizer  $N$  of a maximal torus and to the centralizer  $Q$  of a principal diagonal, both introduced in Section 2. As an application of the “Sandwich” Theorem, the final proposition presents a “minimal” compact Lie group  $G$  such that the associated extension is *not* split.

Let  $\bar{G}_0$  denote the adjoint group  $G_0/Z_0$ ; it is well-known that the center of  $\bar{G}_0$  is trivial.

**THEOREM 5.1.** *Let  $G$  be a compact Lie group with a nonabelian connected component. Then there exist two surjective homomorphisms*

$$\begin{array}{ccccc} G_s = G_0 \rtimes Z & \xrightarrow{\pi_1} & G & \xrightarrow{\pi_2} & \bar{G} = G/Z_0 \cong \bar{G}_0 \rtimes \Gamma, \\ (g_0, z) & \longmapsto & g_0 \cdot z & & \end{array}$$

where the centralizer  $Z$  of a fixed principal subgroup acts on  $G_0$  by conjugation, and where  $\pi_2$  is the canonical projection corresponding to the normal subgroup  $Z_0$  of  $G$ .

For the kernels, we have  $\ker \pi_1 \cong Z_0$  and  $\ker \pi_2 = Z_0$ ; in particular

$$G \cong (G_0 \rtimes Z)/Z_0 \quad \text{and} \quad G/Z_0 \cong \bar{G}_0 \rtimes \Gamma.$$

*Proof.* As  $\bar{G}_0$  is centerless,  $\bar{G}$  must be isomorphic to  $\bar{G}_0 \rtimes \Gamma$  by the proof of Proposition 3.1. The other assertions about  $\pi_2$  are clear.

The map  $\pi_1$  is well-defined and surjective (because  $Z$  intersects every component of  $G$ ). Straightforward computations show that it is a homomorphism and that  $\ker \pi_1 \cong Z_0$ .  $\square$

REMARK 5.2. 1) The component of the identity of  $G_s$  is equal to  $G_o$  if and only if  $G_o$  is semisimple.

2) The present version of the "Sandwich" Theorem has the advantage of being more explicit than the one in [14] (the result therein is an existence theorem). Its drawbacks are the fact that  $Z$  is not finite if  $G_o$  is not semisimple, and that it obviously makes no sense for compact Lie groups with an abelian connected component.

3) Given a homomorphism  $\varphi: \Gamma \rightarrow \text{Out}(G_o)$ , and an extension  $Z_o \xrightarrow{\mu} E \xrightarrow{\nu} \Gamma$  for which the action coincides with the "restriction"  $\bar{\varphi}$ , there is a more direct way than the cohomological one to recover the corresponding compact Lie group  $G$ , i.e. the one that fits into the commutative diagram

$$\begin{array}{ccccc} Z_o & \hookrightarrow & E & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \parallel \\ G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \end{array}$$

Let us define the composition  $\bar{\sigma}: E \xrightarrow{\nu} \Gamma \xrightarrow{\varphi} \text{Out}(G_o) \xrightarrow{s} \text{Aut}(G_o)$ , where  $s$  is as in Theorem 2.4. Then by Bourbaki (see [4], Lemme 7, pp. 210–211), we have

$$G = (G_o \rtimes_{\bar{\sigma}} E) / \Delta Z_o,$$

where  $\Delta Z_o$  is the image of the injection  $z_o \mapsto (z_o^{-1}, \mu(z_o))$ . Taking  $E = Z$ , this gives another proof of the assertions concerning  $\pi_1$  in Theorem 5.1.

Using Cartan subgroups (in the sense of Segal [22], i.e. those Adams called "SS subgroups" in honour of Segal and de Siebenthal [1]), de Siebenthal gave some explicit sufficient conditions for the splitting of the extension associated to  $G$  ([9], Théorème p. 74).

THEOREM 5.3 (de Siebenthal). *Let  $G$  be a compact Lie group with  $G_o$  simply connected, or of adjoint type (i.e.  $Z_o$  is trivial). If  $\Gamma = \pi_o(G)$  is cyclic then  $G$  is a semidirect product, i.e.  $G \cong G_o \rtimes \Gamma$ .*

A relationship with the splitting of the extension associated to the normalizer of a maximal torus  $N$  in  $G$  is given in the next proposition.

PROPOSITION 5.4. *If the group of components  $\Gamma$  of  $G$  is nilpotent, then the extension  $G_o \hookrightarrow G \twoheadrightarrow \Gamma$  is split if and only if the extension  $N_o \hookrightarrow N \twoheadrightarrow \Gamma$  is split.*

*Proof.* The “if” part is clear. Conversely, let  $s: \Gamma \rightarrow G$  be a section. By a result in Bourbaki (see [5], Corollaire 4, p. 49), any nilpotent subgroup of a compact Lie group is contained in the normalizer of some maximal torus. Therefore, if needed after conjugation by an element in  $G_0$ , we have  $s(\Gamma) \subset N$ , and we can conclude that the extension associated to  $N$  is split.  $\square$

For an extended maximal torus  $Q$ , the extensions can be related as follows.

**PROPOSITION 5.5.** *If the group of components  $\Gamma$  of  $G$  is cyclic, then the extension  $G_0 \hookrightarrow G \twoheadrightarrow \Gamma$  is split if and only if the extension  $T \hookrightarrow Q \twoheadrightarrow \Gamma$  is split.*

*Proof.* The proposition readily follows from the fact that the conjugates of  $Q$  cover  $G$ .  $\square$

**REMARK 5.6.** This latter proposition fails in general. An obstruction to the splitting of the extension associated to the extended maximal torus  $Q$  can be found in a paper by Oliver; this obstruction involves the representation ring of  $G$  and its relation with the family of all  $p$ -toral subgroups of  $G$  (see [19], Corollary 3.11). In particular, Oliver constructs a compact Lie group  $G = \mathrm{SU}(2) \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/3)$  such that the extension corresponding to the extended maximal torus  $Q$  is *not* split ([19], pp. 376–377).

We conclude with the promised example.

**PROPOSITION 5.7.** *Let  $D_8 = \langle r, s \mid r^4 = s^2 = e, srs^{-1} = r^{-1} \rangle$  be a presentation of the dihedral group. Then the quotient*

$$G = (\mathrm{SU}(2) \times D_8) / \Delta \mathbf{Z}/2,$$

*where  $\Delta \mathbf{Z}/2$  denotes the central subgroup generated by  $(-\mathbf{1}, r^2)$ , is a compact Lie group with  $G_0 \cong \mathrm{SU}(2)$  and  $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ , the 4-group of Klein. The associated extension*

$$\mathrm{SU}(2) \hookrightarrow G \twoheadrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

*is not split. Among the extensions associated to a compact Lie group with a nonabelian connected component, it is a minimal one having the property of being non-split, in the sense that the rank of the connected component and the size of the group of components are minimal. Moreover replacing the*

connected component by  $\mathrm{SO}(3)$  (i.e. by “any” group of the same rank), or the group of components by  $\mathbf{Z}/4$  (i.e. by “any” group of the same size), will force the extension to be split.

*Proof.* The assertions about the connected component and the group of components are clear. Let us show that the extension associated to  $G$  is not split. Let us denote  $[g, \gamma] \in G$  the image of  $(g, \gamma) \in \mathrm{SU}(2) \times D_8$  under the canonical projection. Let  $\mathbf{S}^1$  denote the standard maximal torus in  $\mathrm{SU}(2)$ , and let  $N$  denote its normalizer in  $G$ . We have

$$N = \{[t, e] : t \in \mathbf{S}^1\} \amalg \{[jt, e] : t \in \mathbf{S}^1\} \amalg \{[t, r] : t \in \mathbf{S}^1\} \amalg \{[jt, r] : t \in \mathbf{S}^1\} \\ \amalg \{[t, s] : t \in \mathbf{S}^1\} \amalg \{[jt, s] : t \in \mathbf{S}^1\} \amalg \{[t, rs] : t \in \mathbf{S}^1\} \amalg \{[jt, rs] : t \in \mathbf{S}^1\}.$$

By contradiction, suppose that the extension associated to  $G$  is split, i.e. there exists a section. As  $\mathbf{Z}/2 \times \mathbf{Z}/2$  is abelian, thus nilpotent, we deduce, by Proposition 5.4, that the extension associated to  $N$  is also split. We want to show that this is not possible by considering the elements of order 2 in  $N$ . For  $n = 0, 1$ , a straightforward calculation shows that in the component corresponding to  $r^n s$ , an element  $[t, r^n s]$  is of order 2 if and only if  $t = \pm 1$ , and that the sub-component  $\{[jt, r^n s] : t \in \mathbf{S}^1\}$  does not contain any element of order 2. Two of the three non-trivial elements in  $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$  must thus be mapped by the section to  $[\pm 1, s]$  and  $[\pm 1, rs]$ . Therefore, as the section is a homomorphism, the image of the third non-trivial element is

$$[\pm 1, rs] \cdot [\pm 1, s] = [\pm 1, r],$$

which is not of order 2. A contradiction that shows that the extension associated to  $G$  is not split.

The property of minimality follows by Theorem 5.3, and by the fact that any extension with  $\mathrm{SO}(3)$  as normal subgroup is a direct product (because  $\mathrm{SO}(3)$  is complete, i.e. centerless and with trivial outer automorphism group).  $\square$

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