2. Compact Lie groups :a review

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is shown that centralizers of principal subgroups are extensions of Γ by the center Z_0 of G_0 that completely control the situation. The Main Theorem is proved in Section 4 and as an illustration, two examples are then given. The final section relates the approach taken in the present work with the natural question of the splitting of the extension associated to a compact Lie group. As an application of principal subgroups, a revisited "Sandwich" Theorem is proved. Particular cases where the extension is always split are also described, and, finally, a "minimal" extension failing to be split is exhibited.

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2. COMPACT LIE GROUPS: A REVIEW

In this section, we recall the existence of subgroups of G whose related extensions have close relationships to that corresponding to G. First, we introduce more notation. Let T be a fixed maximal torus in G_0, and let LT denote its Lie algebra. Let B be a basis of the root system $R = R(G_0, T)$ of G_0 associated to T. Let $\mathcal{H}$ denote the maximal semisimple ideal of the Lie algebra LG_0 of G_0; the *principal diagonal* of G_0 with respect to B is the 1-dimensional subspace given by $D(B) = \{X \in LT : \alpha(X) = \beta(X), \text{ for all } \alpha, \beta \in B\} \cap \mathcal{H} \subset LT$. The image $\Delta = \Delta(B) = \exp(D(B))$ of this subspace under the exponential map is easily seen to be a closed subgroup of T, isomorphic to the circle group $\mathbf{S}^1$. With a slight abuse of language, we will also call this subgroup a *principal diagonal*. We are now ready to recall the definition of one of the key notions of the present work.

DEFINITION 2.1 (de Siebenthal). A *principal subgroup* of G_0 (associated to T) is a connected closed subgroup H such that H is not contained in any proper connected closed subgroup of maximal rank, and such that $\Delta(B) \subset H$ for some basis B of R.

The work of de Siebenthal shows that any compact Lie group possesses a principal subgroup of rank 1, thus isomorphic to SU(2) or SO(3), and that two such principal subgroups are conjugate [8].

NOTE. For the rest of this work, "principal subgroup" will always mean principal subgroup *of rank* 1.

Before stating the main result of this section, which is a direct consequence of the results of de Siebenthal, we introduce three subgroups of G. Let H_T be a principal subgroup associated to T, and let $Z = Z_G(H_T)$ denote its centralizer in G. This subgroup will play a crucial role in the paper. Let also $N = N_G(T)$ be the normalizer of T in G. We will use the convenient notation $N_o = N_{G_o}(T)$ for the intersection of N with G_o, but one should not be confused, N_o is *not* connected (its group of components being the Weyl group of G_o). Finally, we will consider the centralizer $Q = Z_G(\Delta)$ in G of a principal diagonal Δ.

THEOREM 2.2. *For any compact Lie group G there exists a commutative diagram*

$$\begin{tikzcd}
Z_o \arrow[r, hook] \arrow[d, hook] & Z \arrow[r, two heads] \arrow[d, hook] & \Gamma \arrow[d, equal] \\
T \arrow[r, hook] \arrow[d, hook] & Q \arrow[r, two heads] \arrow[d, hook] & \Gamma \arrow[d, equal] \\
N_o \arrow[r, hook] \arrow[d, hook] & N \arrow[r, two heads] \arrow[d, hook] & \Gamma \arrow[d, equal] \\
G_o \arrow[r, hook] & G \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

where each row is a group extension.

Proof. The centralizer of H_T in G_o is equal to the center Z_o (by a theorem of Borel and de Siebenthal [3], this property characterizes the closed subgroups of G_o that are not contained in any proper connected closed subgroup of maximal rank [5, Ex. 15, p. 116]). As Z intersects every component of G ([8], Théorème 4, pp. 253–254), we get an extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$. The other statements are deduced from the fact that $\Delta \subset T$ contains a regular element, i.e. an element that is contained in exactly one maximal torus, namely T in the present case (see [12] or [17] for more details). $\square$

REMARK 2.3. We call the subgroup Q an *extended maximal torus* of G. These subgroups share some important properties with maximal tori: they are all conjugate, and fixing one of them, its conjugates by the elements of G_o cover the whole group G. They appear in the literature under various disguises (see for instance Oliver [19], Section 1, and Segal [22], §4), as explained in [12].

In the final section, we will see how the splitting of the extension associated to G is related to the splitting of the extensions associated to N and Q that appear in Theorem 2.2.

We end this section by recalling a very important result relating the inner, "usual", and outer automorphism groups of a connected compact Lie group. This result is one of the main reasons why the case of compact Lie groups is well controlled when applying the theory of group extensions, as we will see in Section 3. For the proof, we refer to de Siebenthal [9, Théorème, pp. 46–47] (for another approach consult Bourbaki [5], §4.10).

THEOREM 2.4 (de Siebenthal). *Let G_0 be a connected compact Lie group and let $H \subset G_0$ be a principal subgroup. Then the extension*

$$\mathrm{Inn}(G_0) \overset{\iota}{\hookrightarrow} \mathrm{Aut}(G_0) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(G_0)$$

is split, i.e.

$$\mathrm{Aut}(G_0) \cong \mathrm{Inn}(G_0) \rtimes \mathrm{Out}(G_0)\,.$$

A possible splitting is given by $s\colon \mathrm{Out}(G_0) \to \mathrm{Aut}(G_0)$, where $s(\alpha)$ is the unique automorphism in $\pi^{-1}(\alpha)$ fixing H pointwise.

REMARK 2.5. The fact that the extension associated to $\mathrm{Aut}(G_0)$ is split was known before the work of de Siebenthal, at least in the semisimple case, and appeared in a paper of Dynkin [10].

3. COMPACT LIE GROUPS AND EXTENSIONS

We assume knowledge of the classical relationship between group extensions and related cohomology groups of low degree, as first introduced by Eilenberg and Mac Lane [11]. For readers not familiar with it, the textbooks by Mac Lane [16], Robinson [21], or Adem-Milgram [2], provide a thorough treatment; a more concise approach can be found in Kirillov's book [15], and a sketch in Brown's [6]. We now want to apply this relationship to the case of compact Lie groups. We fix a nonabelian connected compact Lie group G_0, a finite group Γ, and a homomorphism $\varphi\colon \Gamma \to \mathrm{Out}(G_0)$. Recall that Z_0 denotes the center of G_0. Choosing a principal subgroup $H \subset G_0$ and fixing s as in Theorem 2.4, we get the commutative diagram