

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	49 (2003)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
<b>Autor:</b>	Etingof, Pavel / Strickland, Elisabetta
<b>Kapitel:</b>	3.5 The spherical subalgebra
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-66677">https://doi.org/10.5169/seals-66677</a>

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 15.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

*Proof.* It is easy to see that the map  $\mu$  is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials  $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} w$  are linearly independent in  $H_c$ . To do this, it suffices to show that the images of these monomials under the homomorphism  $\phi$ , i.e.  $x_1^{i_1} \dots x_n^{i_n} D_{x_1}^{j_1} \dots D_{x_n}^{j_n} w$ , are linearly independent.

Given an element  $A \in \mathcal{A}$ , writing  $A = \sum_{w \in W} P_w w$  with  $P_w \in \mathcal{D}(U)$  we define the order of  $A$ ,  $\text{ord}A$ , as the maximum of the orders of the  $P_w$ 's. Notice that  $\text{ord}AB \leq \text{ord}A + \text{ord}B$ . We now remark that for any sequence of non negative indices  $(i_1, \dots, i_n)$ ,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + \text{l.o.t.}$$

Indeed this is true for  $D_{x_i}$ . We proceed by induction on  $r = i_1 + \dots + i_n$ . We can clearly assume  $i_1 > 0$ , so by induction,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = (\partial_{x_1} + \text{l.o.t.})(\partial_{x_1}^{i_1-1} \dots \partial_{x_n}^{i_n} + \text{l.o.t.}) = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + \text{l.o.t.}$$

From this we deduce that for any pair of multiindices  $I = (i_1, \dots, i_n)$ ,  $J = (j_1, \dots, j_n)$ ,  $w \in W$ , setting  $x_I = x_1^{i_1} \dots x_n^{i_n}$ ,  $D_J = D_{x_1}^{j_1} \dots D_{x_n}^{j_n}$ ,  $\partial_J = \partial_{x_1}^{j_1} \dots \partial_{x_n}^{j_n}$ , we have

$$x_I D_J w = x_I \partial_J w + \text{l.o.t.}$$

Using this and the linear independence of the elements  $x_I \partial_J w$ , it is immediate to conclude that the elements  $x_I D_J w$  are linearly independent, proving our claim.  $\square$

**REMARK 1.** We see that the homomorphism  $\phi$  identifies  $H_c$  with the subalgebra of  $\mathcal{A}$  generated by  $\mathbf{C}[\mathfrak{h}]$ , the Dunkl operators  $D_y$ ,  $y \in \mathfrak{h}$  and  $W$ .

**REMARK 2.** Another way to state the PBW theorem is the following. Let  $F^\bullet$  be a filtration on  $H_c$  defined by  $\deg(x_i) = \deg(y_i) = 1$ ,  $\deg(w) = 0$ . Then we have a natural surjective mapping from  $\mathbf{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$  to the associated graded algebra  $\text{gr}(H_c)$ . The PBW theorem claims that this map is in fact an isomorphism.

### 3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e = \frac{1}{W} \sum_{w \in W} w \in \mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of  $H_c$  is the algebra  $eH_ce$ .

Notice that  $1 \notin eH_ce$ . On the other hand, since  $ex = xe = e$  for  $x \in eH_ce$ ,  $e$  is the unit for the spherical subalgebra. We can embed both  $\mathbf{C}[\mathfrak{h}^*]^W$  and  $\mathbf{C}[\mathfrak{h}]^W$  in the spherical subalgebra as follows. Take  $f \in \mathbf{C}[\mathfrak{h}^*]^W$  (the other case is identical) and set  $m_e(f) = fe$ . Since  $f$  is invariant, we have  $efe = fe^2 = fe = m_e(f)$ , so that  $m_e$  actually maps  $\mathbf{C}[\mathfrak{h}^*]^W$  to  $eH_ce$ . The injectivity is clear from the PBW-theorem. As for the fact that  $m_e$  is a homomorphism, we have  $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$ . From now on, we will consider both  $\mathbf{C}[\mathfrak{h}^*]^W$  and  $\mathbf{C}[\mathfrak{h}]^W$  as subalgebras of the spherical subalgebra.

### 3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras  $H_c$  and  $eH_ce$ .

DEFINITION 3.14. The category  $\mathcal{O}(H_c)$  (resp.  $\mathcal{O}(eH_ce)$ ) is the full subcategory of the category of  $H_c$ -modules (resp.  $eH_ce$ -modules) whose objects are the modules  $M$  such that

- 1)  $M$  is finitely generated.
- 2) For all  $v \in M$ , the subspace  $\mathbf{C}[\mathfrak{h}^*]^Wv \subset M$  is finite dimensional.

We can define a functor

$$F: \mathcal{O}(H_c) \rightarrow \mathcal{O}(eH_ce)$$

by setting  $F(M) = eM$ . It is easy to show that  $F(M)$  is an object of  $\mathcal{O}(eH_ce)$ .

We are now going to explain how to construct some modules in  $\mathcal{O}(H_c)$  which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take  $\lambda \in \mathfrak{h}^*$ . Denote by  $W_\lambda \subset W$  the stabilizer of  $\lambda$ . Take an irreducible  $W_\lambda$ -module  $\tau$ . We define a structure of  $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$ -module on  $\tau$  by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau, w \in W_\lambda, f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by  $\lambda\#\tau$ . We can then consider the  $H_c$ -module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda\#\tau.$$

This is called a Whittaker module. In the special case  $\lambda = 0$  (and hence  $W_\lambda = W$ ), the module  $M(0, \tau)$  is called a Verma module. It is clear that these are objects of  $\mathcal{O}$ . Notice that as  $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$ -module,  $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$ .