

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor: Etingof, Pavel / Strickland, Elisabetta
Kapitel: 3.2 Berest's formula for L_q
DOI: <https://doi.org/10.5169/seals-66677>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 12.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

differential operator S_m of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x) = \prod_{s \in \Sigma} \alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q \in \mathbf{C}[\mathfrak{h}] = \mathbf{C}[q_1, \dots, q_n]$. From this, if we set $\psi(k, x) = S_m e^{(k, x)}$, we get

$$(7) \quad L_q \psi = S_m q(\partial) e^{(k, x)} = q(k) \psi,$$

$q \in \mathbf{C}[q_1, \dots, q_n]$.

We claim that equation (7) must in fact hold for all $q \in Q_m$. Indeed, near a generic point x , the functions $\psi(wk, x)$ are obviously linearly independent and satisfy (7) for symmetric q . Thus, they are a basis in the space of solutions (we know that this space is $|W|$ -dimensional). Consider the matrix of L_q in this basis for any $q \in Q_m$. Since $\psi(k, x)$ is a polynomial multiplied by $e^{(k, x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$,

$$S_m = (x\partial - 2m + 1)(x\partial - 2m - 1) \cdots (x\partial - 1).$$

3.2 BEREST'S FORMULA FOR L_q

We are now going to give an explicit construction of the operators L_q for any $q \in Q_m$.

Let us identify, using our W -invariant scalar product, \mathfrak{h} with \mathfrak{h}^* , and let us choose a orthonormal basis x_1, \dots, x_n in \mathfrak{h}^* . If $x \in \mathfrak{h}^*$, we will write D_x for the Dunkl operator relative to the vector in \mathfrak{h} corresponding to x under our identification. Thus

$$L = \sum_{i=1}^n D_{x_i}^2.$$

PROPOSITION 3.2 (Berest [Be]). *If $q \in Q_m$ is a homogeneous element of degree d , then*

$$(\text{ad } L)^{d+1} q = 0.$$

Proof. It is enough to prove that

$$((\text{ad } L)^{d+1} q) \psi(k, x) = 0.$$

Indeed, it follows from the definition of $\psi(k, x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\text{ad } L)^{d+1} q) S_m = 0$, so that $(\text{ad } L)^{d+1} q = 0$, since $\mathcal{D}(U)$ is a domain.

Given $q \in Q_m$, we will denote by $L_q^{(k)}$ the operator $q(D_{k_1}, \dots, D_{k_n})$. Notice that since $\psi(k, x) = \psi(x, k)$, we have $L_q^{(k)}\psi = q(x)\psi$. Thus we deduce, for $p, q, r \in Q_m$,

$$\begin{aligned} L_q r(x) L_p \psi &= L_q r(x) p(k) \psi = p(k) L_q r(x) \psi \\ &= p(k) L_q L_r^{(k)} \psi = p(k) L_r^{(k)} L_q \psi = p(k) L_r^{(k)} q(k) \psi. \end{aligned}$$

It follows that

$$(\text{ad } L)^{d+1} q \psi = (-1)^{d+1} (\text{ad}(\sum_{i=1}^n k_i^2))^{d+1} L_q^{(k)} \psi.$$

Since L_q is a differential operator of degree d , we get $\text{ad}(\sum_{i=1}^n k_i^2)^{d+1} L_q^{(k)} = 0$, as desired. \square

Notice now that the operator $(\text{ad } L)^d q(x)$ commutes with L . Its symbol is given by $(\text{ad } \Delta)^d q(x) = 2^d d! q(\partial)$. So we deduce the following

COROLLARY 3.3 (Berest's formula, [Be]). *If $q \in Q_m$ is homogeneous of degree d , then*

$$L_q = \frac{1}{2^d d!} (\text{ad } L)^d q(x).$$

Proof. This is clear from Proposition 2.8, once we remark that $(\text{ad } L)^d q(x)$ has the required homogeneity. \square

We want to give a representation theoretical interpretation of what we have just seen. Consider the three operators

$$(8) \quad F = \frac{\sum_{i=1}^n x_i^2}{2}, \quad E = -\frac{L}{2}, \quad H = [E, F].$$

It is easy to check that $[H, E] = 2E$, $[H, F] = -2F$. We deduce that the elements E, F, H span an $\mathfrak{sl}(2)$ Lie subalgebra of $\mathcal{D}(U)$. Thus $\mathfrak{sl}(2)$ acts by conjugation on $\mathcal{D}(U)$. We can then reformulate Proposition 3.2 as follows:

PROPOSITION 3.4. *Any polynomial $q \in Q_m$ of degree d is a lowest weight vector for the $\mathfrak{sl}(2)$ -action of weight $-d$ and generates a finite dimensional module (necessarily of dimension $d+1$) for which L_q is a highest weight vector.*

Proof. An easy direct computation shows that

$$H = [E, F] = - \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} + C,$$

where C is a constant. Thus if q is homogeneous of degree d , we have $[H, L_q] = dL_q$.

This and the fact that $[L, L_q] = 0$, implies that L_q is a highest weight vector of weight d . Also since F is a polynomial, we deduce that $\text{ad } F^{d+1} L_q = 0$, so that L_q generates a $(d+1)$ -dimensional irreducible $\mathfrak{sl}(2)$ -module. \square

One last property about these operators is given by

PROPOSITION 3.5 ([FV]). *For any $q \in Q_m$, the operator L_q preserves Q_m .*

Proof. Let us begin by proving that L preserves Q_m .

Take $f \in Q_m$, so that for any $s \in \Sigma$, $f - {}^s f = \alpha_s^{2m_s+1} t$, $t \in \mathbf{C}[h]$. Let us start by showing that Lf is a polynomial. Clearly $Lf = \delta_*^{-1} q$, with $q \in \mathbf{C}[h]$, and $\delta_* = \prod_{s: m_s \neq 0} \alpha_s$. Since L is W -invariant, $Lf - {}^s(Lf) = L(f - {}^s f)$ is clearly divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In particular, it is always regular along the reflection hyperplane of s . On the other hand, since $Lf - {}^s(Lf) = \delta_*^{-1}(q + {}^s q)$, we deduce that $q + {}^s q$ is divisible by α_s if $m_s > 0$. But then $q = ((q + {}^s q) + (q - {}^s q))/2$ is divisible by α_s if $m_s > 0$, hence it is divisible by δ_* , so that Lf lies in $\mathbf{C}[h]$.

We have already remarked that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In fact

$$L(f - {}^s f) = (L\alpha_s^{2m_s+1})t + \alpha_s^{2m_s} \tilde{t},$$

where \tilde{t} is a suitable polynomial.

But since

$$\begin{aligned} L\alpha_s^{2m_s+1} &= 2m_s(2m_s+1)(\alpha_s, \alpha_s)\alpha_s^{2m_s-1} - 2m_{s'}(2m_s+1) \sum_{s' \in \Sigma} (\alpha_{s'}, \alpha_s) \frac{\alpha_s^{2m_s}}{\alpha_{s'}} \\ &= -2m_{s'}(2m_s+1) \sum_{s' \in \Sigma, s' \neq s} (\alpha_{s'}, \alpha_s) \frac{\alpha_s^{2m_s}}{\alpha_{s'}}, \end{aligned}$$

we deduce that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s}$. On the other hand, since $L(f - {}^s f) = Lf - {}^s(Lf)$, this polynomial is either zero or it must vanish to odd order on the reflection hyperplane of s . We deduce that it must be divisible by $\alpha_s^{2m_s+1}$, proving that $Lf \in Q_m$.

We now pass to a general L_q , $q \in Q_m$. We may assume that q is homogeneous of, say, degree d . By Corollary 3.3 we have that L_q is a non zero multiple of $(adL)^d(q)$. Since both q and L preserve Q_m , our claim follows. \square

3.3 DIFFERENTIAL OPERATORS ON X_m

Now let us return to the algebra of differential operators $\mathcal{D}(X_m)$. Notice that $\mathcal{D}(X_m)$ contains two commutative subalgebras (both isomorphic to Q_m). The first is Q_m itself, the second is the subalgebra Q_m^\dagger consisting of the differential operators of the form L_q with $q \in Q_m$. It is possible to prove

THEOREM 3.6 ([BEG]). $\mathcal{D}(X_m)$ is generated by Q_m and Q_m^\dagger .

Notice that by Corollary 3.3 we in fact have that $\mathcal{D}(X_m)$ is generated by Q_m and by L .

EXAMPLE 3.7. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$ we get that $\mathcal{D}(X_m)$ is generated by the operators

$$x^2, \quad x^{2m+1}, \quad \frac{d^2}{dx^2} - \frac{2m}{x} \frac{d}{dx}.$$

Theorem 3.6 together with Proposition 3.4, imply

COROLLARY 3.8 ([BEG]). $\mathcal{D}(X_m)$ is locally finite dimensional under the action of the Lie algebra $\mathfrak{sl}(2)$ defined in (8).

This Corollary implies that our $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$ can be integrated to an action of the group $SL(2)$. In particular we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q = L_q$$

for all $q \in Q_m$. This transformation is a generalization of the Fourier transform, since it reduces to the usual Fourier transform on differential operators on \mathfrak{h} when $m = 0$.

EXAMPLE 3.9. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, we get that the monomials $\{x^{2i}\} \cup \{x^{2i+2m+1}\}$ are (up to constants) all lowest weight vectors for the $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$. x^n has weight $-n$. We deduce that $\mathcal{D}(X_m)$ is isomorphic as a $\mathfrak{sl}(2)$ -module to the direct sum of the irreducible representations of dimension $n+1$ for n even or $n = 2(m+i)+1$, each with multiplicity one.