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The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi, \\ \psi'' - \left(\frac{1}{x} + k^2x\right)\psi' = -k^3x\psi. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2x}{kx - 1}\psi,$$

whose solution (up to constants) is given by $\psi = (kx - 1)e^{kx}$.

In fact, one can easily calculate ψ_m for a general m .

PROPOSITION 2.12. $\psi_m(k, x) = (x\partial - 2m + 1)(x\partial - 2m - 1) \cdots (x\partial - 1)e^{kx}$.

Proof. We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$\left(\partial^2 - \frac{2m}{x}\partial\right)(x\partial - 2m + 1) = (x\partial - 2m + 1)\left(\partial^2 - \frac{2(m-1)}{x}\partial\right)$$

as it is easy to verify directly. So using induction on m starting with $m = 0$, we get

$$\left(\partial^2 - \frac{2m}{x}\partial\right)\psi_m(k, x) = (x\partial - 2m + 1)\left(\partial^2 - \frac{2(m-1)}{x}\partial\right)\psi_{m-1}(k, x) = k^2\psi_m(k, x),$$

and $\psi_m(k, x)$ is our solution. \square

3. LECTURE 3

3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k, x)$ for the operator

$$L = \Delta - \sum_{s \in \Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s}.$$

The way to construct $\psi(k, x)$ is via the Opdam shift operator. Given a function $m: \Sigma \rightarrow \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique W -invariant

differential operator S_m of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x) = \prod_{s \in \Sigma} \alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q \in \mathbf{C}[\mathfrak{h}] = \mathbf{C}[q_1, \dots, q_n]$. From this, if we set $\psi(k, x) = S_m e^{(k, x)}$, we get

$$(7) \quad L_q \psi = S_m q(\partial) e^{(k, x)} = q(k) \psi,$$

$q \in \mathbf{C}[q_1, \dots, q_n]$.

We claim that equation (7) must in fact hold for all $q \in Q_m$. Indeed, near a generic point x , the functions $\psi(wk, x)$ are obviously linearly independent and satisfy (7) for symmetric q . Thus, they are a basis in the space of solutions (we know that this space is $|W|$ -dimensional). Consider the matrix of L_q in this basis for any $q \in Q_m$. Since $\psi(k, x)$ is a polynomial multiplied by $e^{(k, x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$,

$$S_m = (x\partial - 2m + 1)(x\partial - 2m - 1) \cdots (x\partial - 1).$$

3.2 BEREST'S FORMULA FOR L_q

We are now going to give an explicit construction of the operators L_q for any $q \in Q_m$.

Let us identify, using our W -invariant scalar product, \mathfrak{h} with \mathfrak{h}^* , and let us choose an orthonormal basis x_1, \dots, x_n in \mathfrak{h}^* . If $x \in \mathfrak{h}^*$, we will write D_x for the Dunkl operator relative to the vector in \mathfrak{h} corresponding to x under our identification. Thus

$$L = \sum_{i=1}^n D_{x_i}^2.$$

PROPOSITION 3.2 (Berest [Be]). *If $q \in Q_m$ is a homogeneous element of degree d , then*

$$(\text{ad } L)^{d+1} q = 0.$$

Proof. It is enough to prove that

$$((\text{ad } L)^{d+1} q) \psi(k, x) = 0.$$

Indeed, it follows from the definition of $\psi(k, x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\text{ad } L)^{d+1} q) S_m = 0$, so that $(\text{ad } L)^{d+1} q = 0$, since $\mathcal{D}(U)$ is a domain.