

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	49 (2003)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
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<b>Kapitel:</b>	2.3 The quantum Calogero-Moser System
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-66677">https://doi.org/10.5169/seals-66677</a>

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The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group  $W$  generated by reflections acting on the space  $\mathfrak{h}$ , and keep the notation of the previous section. Fix a  $W$ -invariant nondegenerate scalar product  $(-, -)$  on  $\mathfrak{h}$ . It determines a scalar product on  $\mathfrak{h}^*$ . Define the “energy function”

$$E(x, p) = \frac{(p, p)}{2} + \frac{1}{2} \sum_{s \in \Sigma} \frac{\gamma_s(\alpha_s, \alpha_s)}{\alpha_s(x)^2}, \quad x \in \mathfrak{h}, \quad p \in \mathfrak{h}^*$$

on  $T^*\mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$ , where  $\gamma: \Sigma \rightarrow \mathbf{C}$  is a  $W$ -invariant function. Notice that although  $\alpha_s$  is defined up to a non zero constant, by homogeneity,  $E$  is independent of the choice of  $\alpha_s$ . We will call the system defined by  $E$  the Calogero-Moser system for  $W$ .

If  $W$  is the symmetric group  $S_n$ ,  $\mathfrak{h} = \mathbf{C}^n$ , then  $\Sigma$  is the set of transpositions  $s_{i,j}$ ,  $i < j$ , and we can take  $\alpha_s = e_i - e_j$ . Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for  $W$  is completely integrable.

### 2.3 THE QUANTUM CALOGERO-MOSER SYSTEM

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy  $E$  by formally making the substitution

$$p_j \Rightarrow -i\hbar \frac{\partial}{\partial x_j},$$

where  $\hbar$  is a parameter (Planck’s constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2}\Delta + \frac{1}{2} \sum_{s \in \Sigma} \frac{\gamma_s(\alpha_s, \alpha_s)}{\alpha_s^2},$$

where  $\Delta$  denotes the Laplacian.

In particular, in the case of  $W = S_n$  we have

$$\widehat{E} = -\frac{\hbar^2}{2}\Delta + \sum_{i < j} \frac{c}{(x_i - x_j)^2},$$

where  $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ . Setting  $\beta_s = \frac{\gamma_s}{2\hbar^2}$ , we will from now on consider the operator

$$H := -\frac{2}{\hbar^2} \widehat{E} = \Delta - \sum_{s \in \Sigma} \frac{\beta_s(\alpha_s, \alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.

We want to study the stationary Schrödinger equation:

$$(3) \quad H\psi = \lambda\psi, \quad \lambda \in \mathbf{C}.$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator  $H$ , but for the Calogero-Moser operator the situation is much better.

**DEFINITION 2.1.** A *quantum integral* of  $H$  is a differential operator  $M$  such that

$$[M, H] = 0.$$

We are going to show that there are many quantum integrals of  $H$ , namely that there are  $n$  commuting algebraically independent quantum integrals  $M_1, \dots, M_n$  of  $H$ . By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found  $M_1, \dots, M_n$ , observe that for fixed constants  $\mu_1, \dots, \mu_n$ , the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \dots\dots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under  $H$ . We will see that this space is in fact finite dimensional. Therefore, the operators  $M_i$  allow one to reduce the problem of solving the partial differential equation  $H\psi = \lambda\psi$  to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

## 2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS .

We are now going to explain how to find quantum integrals for  $H$ , using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety  $X$ , we will denote by  $\mathcal{D}(X)$  the ring of differential operators on  $X$ . We are going to consider the case in which  $X$  is the open set  $U$  in  $\mathfrak{h}$  which is the complement of the divisor of the equation  $\delta(x) := \prod_{s \in \Sigma} \alpha_s(x)$ . Clearly  $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$ .

**LEMMA 2.2.** *An element of  $\mathcal{D}(U)$  is completely determined by its action on  $\mathbf{C}[U]^W = \mathbf{C}[U/W]$ .*