Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 49 (2003)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA

Autor: Etingof, Pavel / Strickland, Elisabetta

Kapitel: 2.2 The classical Calogero-Moser System

DOI: https://doi.org/10.5169/seals-66677

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 15.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

called states of the system. The dynamics of the system $x = x(t)$, $p = p(t)$ depends on the Hamiltonian, or energy function, $E(x,p)$ on T^*X. Given E and the initial state $x(0)$, $p(0)$, one can recover the dynamics $x = x(t)$, $p = p(t)$ from Hamilton's differential equations $\frac{df(x,p)}{dt} = \{f,E\}$. If X is locally identified with $\mathbf{R}^n$ by choosing coordinates $x_1,\dots,x_n$, then T^*X is locally identified with $\mathbf{R}^{2n}$ with coordinates $x_1,\dots,x_n,p_1,\dots,p_n$. In these coordinates, Hamilton's equations may be written in their standard form

$$\dot{x}_i = \frac{\partial E}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial E}{\partial x_i}.$$

A function $I(x,p)$ is called an integral of motion for our system if $\{I,E\} = 0$. Integrals of motion are useful, since for any such integral I the function $I(x(t),p(t))$ is constant, which allows one to reduce the number of variables by 2. Thus, if we are given n functionally independent integrals of motion $I_1,\dots,I_n$ with $\{I_l,I_k\} = 0$ for all $1 \le l,k \le n$, then all $2n$ variables x_i,p_i can be excluded, and the system can be completely solved by quadratures. Such a situation is called complete (or Liouville) integrability.

2.2 The classical Calogero-Moser system

Quasi-invariants are related to many-particle systems. Consider a system of n particles on the real line $\mathbf{R}$. A potential is an even function

$$U(x) = U(-x), \quad x \in \mathbf{R}.$$

Two particles at points a, b have energy of interaction $U(a-b)$. The total energy of our system of particles is

$$E = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i<j} U(x_i - x_j).$$

Here, x_i are the coordinates of the particles, p_i their momenta. The dynamics of the particles $x_i = x_i(t)$, $p_i = p_i(t)$ is governed by the Hamilton equations with energy function E.

This is a system of nonlinear differential equations, which in general can be difficult to solve explicitly. However, for special potentials this system might be completely integrable. For instance, we will see that this is the case for the Calogero-Moser potential,

$$U(x) = \frac{\gamma}{x^2},$$

γ being a constant.

The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group W generated by reflections acting on the space $\mathfrak{h}$, and keep the notation of the previous section. Fix a W-invariant nondegenerate scalar product $(-,-)$ on $\mathfrak{h}$. It determines a scalar product on $\mathfrak{h}^*$. Define the "energy function"

$$E(x,p) = \frac{(p,p)}{2} + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s(x)^2}, \quad x \in \mathfrak{h}, \quad p \in \mathfrak{h}^*$$

on $T^*\mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$, where $\gamma\colon \Sigma \to \mathbf{C}$ is a W-invariant function. Notice that although α_s is defined up to a non zero constant, by homogeneity, E is independent of the choice of α_s. We will call the system defined by E the Calogero-Moser system for W.

If W is the symmetric group S_n, $\mathfrak{h} = \mathbf{C}^n$, then Σ is the set of transpositions $s_{i,j}$, $i < j$, and we can take $\alpha_s = e_i - e_j$, Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for W is completely integrable.

2.3 The quantum Calogero-Moser system

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy E by formally making the substitution

$$p_j \Rightarrow -i\hbar\frac{\partial}{\partial x_j},$$

where $\hbar$ is a parameter (Planck's constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2}\Delta + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s^2},$$

where Δ denotes the Laplacian.

In particular, in the case of $W = S_n$ we have

$$\widehat{E} = -\frac{\hbar^2}{2}\Delta + \sum_{i<j} \frac{c}{(x_i - x_j)^2},$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Setting $\beta_s = \frac{\gamma_s}{2\hbar^2}$, we will from now on consider the operator

$$H := -\frac{2}{\hbar^2}\widehat{E} = \Delta - \sum_{s\in\Sigma} \frac{\beta_s(\alpha_s,\alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.