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<b>Autor:</b>	Etingof, Pavel / Strickland, Elisabetta
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We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the  $\mathbf{C}[\mathfrak{h}]^W$ -module  $Q_m$  can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category  $\mathcal{O}$ . On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra  $\mathbf{C}[\mathfrak{h}]^W$ .

### 1.5 THE POINCARÉ SERIES OF $Q_m$

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r,$$

where  $Q_m[r]$  denotes the graded component of  $Q_m$  of degree  $r$ . For every irreducible representation  $\tau \in \widehat{W}$ , define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \text{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}][r]) t^r.$$

Consider the element in the group ring  $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s).$$

The  $W$ -invariance of  $m$  implies that  $\mu_m$  lies in the center of  $\mathbf{Z}[W]$ . Hence it is clear that  $\mu_m$  acts as a scalar,  $\xi_m(\tau)$ , on  $\tau$ . Let  $d_\tau$  be the degree of  $\tau$ .

**LEMMA 1.9.** *The scalar  $\xi_m(\tau)$  is an integer.*

*Proof.*  $\mathbf{Z}[W]$  and hence also its center, is a finite  $\mathbf{Z}$ -module. This clearly implies that  $\xi_m(\tau)$  is an algebraic integer. Thus to prove that  $\xi_m(\tau)$  is an integer, it suffices to see that  $\xi_m(\tau)$  is a rational number. Let  $d_{\tau,s}$  be the dimension of the space of  $s$ -invariants in  $\tau$ . Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2m_s (d_\tau - d_{\tau,s}),$$

which gives the rationality of  $\xi_m(\tau)$ .  $\square$

THEOREM 1.10. *One has*

$$(1) \quad h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} \chi_\tau(t).$$

REMARK. This theorem was proved in [FeV] modulo Theorem 1.7 (conjectured in [FV]) using the so-called Matsuo-Cherednik correspondence (see [FeV] for details). Thus, Theorem 1.10 follows from [FeV] and [EG2]. Another proof of this theorem is given in [BEG]; this is the proof we will discuss below (in Lecture 3).

EXAMPLE 1.11. If  $m = 0$ , since  $Q_0 = \mathbf{C}[\mathfrak{h}]$ , the theorem says that

$$h_{Q_0}(t) = \frac{1}{(1-t)^n} = \sum_{\tau \in \widehat{W}} d_\tau \chi_\tau(t).$$

Indeed, as a  $W$ -module one has

$$\mathbf{C}[\mathfrak{h}] = \bigoplus_{\tau} \tau \otimes \text{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}]).$$

EXAMPLE 1.12. If  $W = \mathbf{Z}/2$ , then  $\widehat{W} = \{+, -\}$ , where  $+$  (respectively  $-$ ) denotes the trivial (respectively the sign) representation. One has

$$\mathbf{C}[x] = \mathbf{C}[x^2] \oplus \mathbf{C}[x^2]x,$$

where  $\mathbf{C}[x^2] = \mathbf{C}[x]^W$  and  $\mathbf{C}[x^2]x$  is the isotypic component of the sign representation. Thus

$$\chi_+(t) = \frac{1}{1-t^2}, \quad \chi_-(t) = \frac{t}{1-t^2},$$

$\mu_m = m(1-s)$ . Thus  $\xi_m(+)=0$ ,  $\xi_m(-)=2m$ . We deduce that

$$h_{Q_m}(t) = \frac{1}{1-t^2} + \frac{t^{2m+1}}{1-t^2},$$

as we already know.

Recall now that as a graded  $W$ -module  $\mathbf{C}[\mathfrak{h}]$  is isomorphic to  $\mathbf{C}[\mathfrak{h}]^W \otimes H$ ,  $H$  being the space of harmonic polynomials. We deduce that the  $\tau$ -isotypic component in  $\mathbf{C}[\mathfrak{h}]$  is isomorphic to  $\mathbf{C}[\mathfrak{h}]^W \otimes H_\tau$ .

Set  $K_\tau(t) = \sum_{r \geq 0} \dim \text{Hom}_W(\tau, H[r]) t^r$ . This is a polynomial, called the Kostka polynomial relative to  $\tau$ . We deduce that

$$(2) \quad \chi_\tau(t) = \frac{K_\tau(t)}{\prod_{i=1}^n (1 - t^{d_i})}.$$

Also, if  $\tau' = \tau \otimes \varepsilon$ ,  $\varepsilon$  being the sign representation, one has

$$K_{\tau'}(t) = K_\tau(t^{-1}) t^{|\Sigma|}.$$

Set now

$$P_m(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} K_\tau(t).$$

We have

PROPOSITION 1.13 ([FeV]).

$$h_{Q_m}(t) = \frac{P_m(t)}{\prod_{i=1}^n (1 - t^{d_i})}.$$

Furthermore  $P_m(t) = t^{\xi_m(\varepsilon) + |\Sigma|} P_m(t^{-1})$ .

*Proof.* Substituting the expression (2) for  $\chi_\tau(t)$  in (1.10) and using the definition of  $P_m(t)$ , we get

$$h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} \frac{K_\tau(t)}{\prod_{i=1}^n (1 - t^{d_i})} = \frac{P_m(t)}{\prod_{i=1}^n (1 - t^{d_i})},$$

as desired.

Now notice that

$$\xi_m(\tau) + \xi_m(\tau') = \sum_{s \in \Sigma} 2m_s = \xi_m(\varepsilon).$$

Using this we get

$$\begin{aligned} t^{\xi_m(\varepsilon) + |\Sigma|} P_m(t^{-1}) &= \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\varepsilon) - \xi_m(\tau)} t^{|\Sigma|} K_\tau(t^{-1}) \\ &= \sum_{\tau' \in \widehat{W}} d_{\tau'} t^{\xi_m(\tau')} K_{\tau'}(t) = P_m(t), \end{aligned}$$

as desired.  $\square$

From this we deduce

**THEOREM 1.14** ([EG2, BEG, FeV], conjectured in [FV]). *The ring  $Q_m$  of  $m$ -quasi-invariants is Gorenstein.*

*Proof.* By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain  $A$  is Gorenstein iff its Poincaré series is a rational function  $h(t)$  satisfying the equation  $h(t^{-1}) = (-1)^n t^l h(t)$ , where  $l$  is an integer and  $n$  is the dimension of the spectrum of  $A$ . Thus the result follows immediately from Proposition 1.13.  $\square$

## 1.6 THE RING OF DIFFERENTIAL OPERATORS ON $X_m$

Finally, let us introduce the ring  $\mathcal{D}(X_m)$  of differential operators on  $X_m$ , that is the ring of differential operators with coefficients in  $\mathbf{C}(\mathfrak{h})$  mapping  $Q_m$  to  $Q_m$ . It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

**THEOREM 1.15** ([BEG]).  *$\mathcal{D}(X_m)$  is a simple algebra.*

**REMARK 1.16.** a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

## 2. LECTURE 2

We will now see how the ring  $Q_m$  appears in the theory of completely integrable systems.

### 2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space  $X$  (a smooth manifold). Then the phase space of this system is  $T^*X$ , the cotangent bundle on  $X$ . The space  $T^*X$  is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on  $T^*X$ . A point of  $T^*X$  is a pair  $(x, p)$ , where  $x \in X$  is the position and  $p \in T_x^*X$  is the momentum. Such pairs are