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Autor: Etingof, Pavel / Strickland, Elisabetta
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W_z -invariant, we deduce that $p(x) - p(sx) = 0$, so that in this case $p(x) - p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z) \neq 0$. Indeed, for a reflection s , α_s vanishes exactly on the fixed points of s , so that $\prod_{s \in \Sigma, sz \neq z} \alpha_s(z)^{2m_s+1} \neq 0$. Also for all $w \in W_z$ $f(wz) = f(z) \neq 0$. On the other hand, it is clear that $p(y) = 0$. \square

EXAMPLE 1.5. Take $W = \mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$. From this we deduce that setting $z = x^2$ and $y = x^{2m+1}$, $Q_m = \mathbf{C}[y, z]/(y^2 - z^{2m+1}) = \mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2 = z^{2m+1}$. The map $\pi: \mathbf{C} \rightarrow K$ is given by $\pi(t) = (t^{2m+1}, t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over \mathbf{C} and $A = \mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1, \dots, f_n \in \mathbf{C}[X]$ which are algebraically independent over \mathbf{C} and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1, \dots, f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist f_1, \dots, f_n as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1, \dots, f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any f_1, \dots, f_n which are algebraically independent over \mathbf{C} and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1, \dots, f_n]$, we have that A is a locally free $\mathbf{C}[f_1, \dots, f_n]$ -module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). Q_m is Cohen-Macaulay.

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$ -module.

We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the $\mathbf{C}[\hbar]^W$ -module Q_m can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category \mathcal{O} . On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra $\mathbf{C}[\hbar]^W$.

1.5 THE POINCARÉ SERIES OF Q_m

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r,$$

where $Q_m[r]$ denotes the graded component of Q_m of degree r . For every irreducible representation $\tau \in \widehat{W}$, define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \operatorname{Hom}_W(\tau, \mathbf{C}[\hbar][r]) t^r.$$

Consider the element in the group ring $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s).$$

The W -invariance of m implies that μ_m lies in the center of $\mathbf{Z}[W]$. Hence it is clear that μ_m acts as a scalar, $\xi_m(\tau)$, on τ . Let d_τ be the degree of τ .

LEMMA 1.9. *The scalar $\xi_m(\tau)$ is an integer.*

Proof. $\mathbf{Z}[W]$ and hence also its center, is a finite \mathbf{Z} -module. This clearly implies that $\xi_m(\tau)$ is an algebraic integer. Thus to prove that $\xi_m(\tau)$ is an integer, it suffices to see that $\xi_m(\tau)$ is a rational number. Let $d_{\tau,s}$ be the dimension of the space of s -invariants in τ . Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2m_s (d_\tau - d_{\tau,s}),$$

which gives the rationality of $\xi_m(\tau)$. \square