

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor: Etingof, Pavel / Strickland, Elisabetta
Kapitel: 1.3 The variety X_m and its bijective normalization
DOI: <https://doi.org/10.5169/seals-66677>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

REMARK. In fact, since W is a finite Coxeter group, a celebrated result of Chevalley says that the algebra $\mathbf{C}[\mathfrak{h}]^W$ is not only a finitely generated \mathbf{C} -algebra but actually a free (=polynomial) algebra. Namely, it is of the form $\mathbf{C}[q_1, \dots, q_n]$, where the q_i are homogeneous polynomials of some degrees d_i . Furthermore, if we denote by H the subspace of $\mathbf{C}[\mathfrak{h}]$ of harmonic polynomials, i.e. of polynomials killed by W -invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \rightarrow \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of $\mathbf{C}[\mathfrak{h}]^W$ - and of W -modules. In particular, $\mathbf{C}[\mathfrak{h}]$ is a free $\mathbf{C}[\mathfrak{h}]^W$ -module of rank $|W|$.

1.3 THE VARIETY X_m AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety $X_m = \text{Spec}(Q_m)$. The inclusion $Q_m \subset \mathbf{C}[\mathfrak{h}]$ induces a morphism

$$\pi: \mathfrak{h} \rightarrow X_m,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that X_m is singular for all $m \neq 0$.)

In fact, not only is π birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]). *π is a bijection.*

Proof. By the above remarks, we only have to show that π is injective. In order to achieve this, we need to prove that quasi-invariants separate points of \mathfrak{h} , i.e. that if $z, y \in \mathfrak{h}$ and $z \neq y$, then there exists $p \in Q_m$ such that $p(z) \neq p(y)$. This is obtained in the following way. Let $W_z \subset W$ be the stabilizer of z and choose $f \in \mathbf{C}[\mathfrak{h}]$ such that $f(z) \neq 0$, $f(y) = 0$. Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx).$$

We claim that $p(x) \in Q_m$. Indeed, let $s \in \Sigma$ and assume that $s(z) \neq z$.

We have by definition $p(x) = \alpha_s(x)^{2m_s+1} \tilde{p}(x)$, with $\tilde{p}(x)$ a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1} \tilde{p}(x) - \alpha_s(sx)^{2m_s+1} \tilde{p}(sx) = \alpha_s(x)^{2m_s+1} (\tilde{p}(x) + \tilde{p}(sx)).$$

If on the other hand, $sz = z$, i.e. $s \in W_z$, then s preserves the set $W \setminus W_z$, and hence preserves $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$ (as it acts by -1 on the products $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$ and $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$). Since $\prod_{w \in W_z} f(wx)$ is

W_z -invariant, we deduce that $p(x) - p(sx) = 0$, so that in this case $p(x) - p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z) \neq 0$. Indeed, for a reflection s , α_s vanishes exactly on the fixed points of s , so that $\prod_{s \in \Sigma, sz \neq z} \alpha_s(z)^{2m_s+1} \neq 0$. Also for all $w \in W_z$ $f(wz) = f(z) \neq 0$. On the other hand, it is clear that $p(y) = 0$. \square

EXAMPLE 1.5. Take $W = \mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$. From this we deduce that setting $z = x^2$ and $y = x^{2m+1}$, $Q_m = \mathbf{C}[y, z]/(y^2 - z^{2m+1}) = \mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2 = z^{2m+1}$. The map $\pi: \mathbf{C} \rightarrow K$ is given by $\pi(t) = (t^{2m+1}, t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over \mathbf{C} and $A = \mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1, \dots, f_n \in \mathbf{C}[X]$ which are algebraically independent over \mathbf{C} and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1, \dots, f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist f_1, \dots, f_n as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1, \dots, f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any f_1, \dots, f_n which are algebraically independent over \mathbf{C} and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1, \dots, f_n]$, we have that A is a locally free $\mathbf{C}[f_1, \dots, f_n]$ -module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). Q_m is Cohen-Macaulay.

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$ -module.