

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	49 (2003)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor:	Etingof, Pavel / Strickland, Elisabetta
Kapitel:	1.2 Elementary properties of Q_m
DOI:	https://doi.org/10.5169/seals-66677

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

EXAMPLE 1.2. The group $W = \mathbf{Z}/2$ acts on $\mathfrak{h} = \mathbf{C}$ by $s(v) = -v$. In this case m is a non negative integer and $\Sigma = \{s\}$. So this definition says that q is in Q_m iff $q(x) - q(-x)$ is divisible by x^{2m+1} . It is very easy to write a basis of Q_m . It is given by the polynomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$.

1.2 ELEMENTARY PROPERTIES OF Q_m

Some elementary properties of Q_m are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

- 1) $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$, $Q_0 = \mathbf{C}[\mathfrak{h}]$, $Q_m \subset Q_{m'}$ if $m \geq m'$, $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$.
- 2) Q_m is a graded subalgebra of $\mathbf{C}[\mathfrak{h}]$.
- 3) The fraction field of Q_m is equal to $\mathbf{C}(\mathfrak{h})$.
- 4) Q_m is a finite $\mathbf{C}[\mathfrak{h}]^W$ -module and a finitely generated algebra. $\mathbf{C}[\mathfrak{h}]$ is a finite Q_m -module.

Proof. 1) is immediate and has already been mentioned in 1.1.

2) Clearly Q_m is closed under addition. Let $p, q \in Q_m$. Let $s \in \Sigma$. Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both $p(x) - p(sx)$ and $q(x) - q(sx)$ are divisible by $\alpha_s^{2m_s+1}$, we deduce that $p(x)q(x) - p(sx)q(sx)$ is also divisible by $\alpha_s^{2m_s+1}$, proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$ for each $s \in \Sigma$, hence $\delta_{2m+1} \in Q_m$. Take $f(x) \in \mathbf{C}[\mathfrak{h}]$. We claim that $f(x)\delta_{2m+1}(x) \in Q_m$. As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition $\delta_{2m+1}(x)$ is divisible by $\alpha_s(x)^{2m_s+1}$ for all $s \in \Sigma$. This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that $\mathbf{C}[\mathfrak{h}]^W$ is a finitely generated algebra over \mathbf{C} and $\mathbf{C}[\mathfrak{h}]$ is a finite $\mathbf{C}[\mathfrak{h}]^W$ -module and hence a finite Q_m -module, proving the second part of 4).

Now $Q_m \subset \mathbf{C}[\mathfrak{h}]$ is a submodule of the finite module $\mathbf{C}[\mathfrak{h}]$ over the Noetherian ring $\mathbf{C}[\mathfrak{h}]^W$. Hence it is finite. This immediately implies that Q_m is a finitely generated algebra over \mathbf{C} . \square

REMARK. In fact, since W is a finite Coxeter group, a celebrated result of Chevalley says that the algebra $\mathbf{C}[\mathfrak{h}]^W$ is not only a finitely generated \mathbf{C} -algebra but actually a free (=polynomial) algebra. Namely, it is of the form $\mathbf{C}[q_1, \dots, q_n]$, where the q_i are homogeneous polynomials of some degrees d_i . Furthermore, if we denote by H the subspace of $\mathbf{C}[\mathfrak{h}]$ of harmonic polynomials, i.e. of polynomials killed by W -invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \rightarrow \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of $\mathbf{C}[\mathfrak{h}]^W$ - and of W -modules. In particular, $\mathbf{C}[\mathfrak{h}]$ is a free $\mathbf{C}[\mathfrak{h}]^W$ -module of rank $|W|$.

1.3 THE VARIETY X_m AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety $X_m = \text{Spec}(Q_m)$. The inclusion $Q_m \subset \mathbf{C}[\mathfrak{h}]$ induces a morphism

$$\pi: \mathfrak{h} \rightarrow X_m,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that X_m is singular for all $m \neq 0$.)

In fact, not only is π birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]). *π is a bijection.*

Proof. By the above remarks, we only have to show that π is injective. In order to achieve this, we need to prove that quasi-invariants separate points of \mathfrak{h} , i.e. that if $z, y \in \mathfrak{h}$ and $z \neq y$, then there exists $p \in Q_m$ such that $p(z) \neq p(y)$. This is obtained in the following way. Let $W_z \subset W$ be the stabilizer of z and choose $f \in \mathbf{C}[\mathfrak{h}]$ such that $f(z) \neq 0$, $f(y) = 0$. Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx).$$

We claim that $p(x) \in Q_m$. Indeed, let $s \in \Sigma$ and assume that $s(z) \neq z$.

We have by definition $p(x) = \alpha_s(x)^{2m_s+1} \tilde{p}(x)$, with $\tilde{p}(x)$ a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1} \tilde{p}(x) - \alpha_s(sx)^{2m_s+1} \tilde{p}(sx) = \alpha_s(x)^{2m_s+1} (\tilde{p}(x) + \tilde{p}(sx)).$$

If on the other hand, $sz = z$, i.e. $s \in W_z$, then s preserves the set $W \setminus W_z$, and hence preserves $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$ (as it acts by -1 on the products $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$ and $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$). Since $\prod_{w \in W_z} f(wx)$ is