

## 1.2 Elementary properties of $\mathbb{Q}_m$

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EXAMPLE 1.2. The group  $W = \mathbf{Z}/2$  acts on  $\mathfrak{h} = \mathbf{C}$  by  $s(v) = -v$ . In this case  $m$  is a non negative integer and  $\Sigma = \{s\}$ . So this definition says that  $q$  is in  $Q_m$  iff  $q(x) - q(-x)$  is divisible by  $x^{2m+1}$ . It is very easy to write a basis of  $Q_m$ . It is given by the polynomials  $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$ .

## 1.2 ELEMENTARY PROPERTIES OF $Q_m$

Some elementary properties of  $Q_m$  are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

- 1)  $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$ ,  $Q_0 = \mathbf{C}[\mathfrak{h}]$ ,  $Q_m \subset Q_{m'}$  if  $m \geq m'$ ,  $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$ .
- 2)  $Q_m$  is a graded subalgebra of  $\mathbf{C}[\mathfrak{h}]$ .
- 3) The fraction field of  $Q_m$  is equal to  $\mathbf{C}(\mathfrak{h})$ .
- 4)  $Q_m$  is a finite  $\mathbf{C}[\mathfrak{h}]^W$ -module and a finitely generated algebra.  $\mathbf{C}[\mathfrak{h}]$  is a finite  $Q_m$ -module.

*Proof.* 1) is immediate and has already been mentioned in 1.1.

2) Clearly  $Q_m$  is closed under addition. Let  $p, q \in Q_m$ . Let  $s \in \Sigma$ . Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both  $p(x) - p(sx)$  and  $q(x) - q(sx)$  are divisible by  $\alpha_s^{2m_s+1}$ , we deduce that  $p(x)q(x) - p(sx)q(sx)$  is also divisible by  $\alpha_s^{2m_s+1}$ , proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has  $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$  for each  $s \in \Sigma$ , hence  $\delta_{2m+1} \in Q_m$ . Take  $f(x) \in \mathbf{C}[\mathfrak{h}]$ . We claim that  $f(x)\delta_{2m+1}(x) \in Q_m$ . As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition  $\delta_{2m+1}(x)$  is divisible by  $\alpha_s(x)^{2m_s+1}$  for all  $s \in \Sigma$ . This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that  $\mathbf{C}[\mathfrak{h}]^W$  is a finitely generated algebra over  $\mathbf{C}$  and  $\mathbf{C}[\mathfrak{h}]$  is a finite  $\mathbf{C}[\mathfrak{h}]^W$ -module and hence a finite  $Q_m$ -module, proving the second part of 4).

Now  $Q_m \subset \mathbf{C}[\mathfrak{h}]$  is a submodule of the finite module  $\mathbf{C}[\mathfrak{h}]$  over the Noetherian ring  $\mathbf{C}[\mathfrak{h}]^W$ . Hence it is finite. This immediately implies that  $Q_m$  is a finitely generated algebra over  $\mathbf{C}$ .  $\square$

REMARK. In fact, since  $W$  is a finite Coxeter group, a celebrated result of Chevalley says that the algebra  $\mathbf{C}[\mathfrak{h}]^W$  is not only a finitely generated  $\mathbf{C}$ -algebra but actually a free (=polynomial) algebra. Namely, it is of the form  $\mathbf{C}[q_1, \dots, q_n]$ , where the  $q_i$  are homogeneous polynomials of some degrees  $d_i$ . Furthermore, if we denote by  $H$  the subspace of  $\mathbf{C}[\mathfrak{h}]$  of harmonic polynomials, i.e. of polynomials killed by  $W$ -invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \rightarrow \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of  $\mathbf{C}[\mathfrak{h}]^W$ - and of  $W$ -modules. In particular,  $\mathbf{C}[\mathfrak{h}]$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module of rank  $|W|$ .

### 1.3 THE VARIETY $X_m$ AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety  $X_m = \text{Spec}(Q_m)$ . The inclusion  $Q_m \subset \mathbf{C}[\mathfrak{h}]$  induces a morphism

$$\pi: \mathfrak{h} \rightarrow X_m,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that  $X_m$  is singular for all  $m \neq 0$ .)

In fact, not only is  $\pi$  birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]).  *$\pi$  is a bijection.*

*Proof.* By the above remarks, we only have to show that  $\pi$  is injective. In order to achieve this, we need to prove that quasi-invariants separate points of  $\mathfrak{h}$ , i.e. that if  $z, y \in \mathfrak{h}$  and  $z \neq y$ , then there exists  $p \in Q_m$  such that  $p(z) \neq p(y)$ . This is obtained in the following way. Let  $W_z \subset W$  be the stabilizer of  $z$  and choose  $f \in \mathbf{C}[\mathfrak{h}]$  such that  $f(z) \neq 0$ ,  $f(y) = 0$ . Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx).$$

We claim that  $p(x) \in Q_m$ . Indeed, let  $s \in \Sigma$  and assume that  $s(z) \neq z$ .

We have by definition  $p(x) = \alpha_s(x)^{2m_s+1} \tilde{p}(x)$ , with  $\tilde{p}(x)$  a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1} \tilde{p}(x) - \alpha_s(sx)^{2m_s+1} \tilde{p}(sx) = \alpha_s(x)^{2m_s+1} (\tilde{p}(x) + \tilde{p}(sx)).$$

If on the other hand,  $sz = z$ , i.e.  $s \in W_z$ , then  $s$  preserves the set  $W \setminus W_z$ , and hence preserves  $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$  (as it acts by  $-1$  on the products  $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$  and  $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$ ). Since  $\prod_{w \in W_z} f(wx)$  is