we obtain

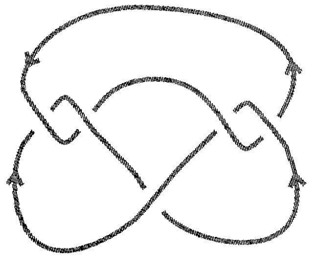
$$\begin{pmatrix} o & e \\ o & o \end{pmatrix} \begin{pmatrix} e & o \\ o & 1 \end{pmatrix} = \begin{pmatrix} e & o \\ o & e \end{pmatrix}.$$

Thus the new u for T_{n+2} is even. Since the connectivity diagram for T_{n+2} in this case, as shown in Figure 36, has compatible standard and palindrome cuts, this result for the parity of u is one step in the verification of the induction hypothesis. Each of the six cases is handled in this same way. We omit the remaining details and assert that the values of u obtained in each case are correct with respect to the connection structure. This completes the proof of Case 2.

Since Cases 1 and 2 encompass all the different possibilities for the standard and palindrome cuts, this completes the proof of the Oriented Schubert Theorem. □

7. STRONGLY INVERTIBLE LINKS

An oriented knot or link is invertible if it is oriented isotopic to the link obtained from it by reversing the orientation of each component. We have seen (Lemma 2) that rational knots and links are invertible. A link L of two components is said to be *strongly invertible* if L is ambient isotopic to itself with the orientation of only one component reversed. In Figure 37 we illustrate the link $L = N([[2],[1],[2]])$. This is a strongly invertible link as is apparent by a 180° vertical rotation. This link is well-known as the Whitehead link, a link with linking number zero. Note that since $[[2],[1],[2]]$ has fraction equal to $2 + 1/(1 + 1/2) = 8/3$ this link is non-trivial via the classification of rational knots and links. Note also that $3 \cdot 3 = 1 + 1 \cdot 8$.



N([[2], [1], [2]]) = W
the Whitehead Link
F(W) = 2+1/(1+1/2) = 8/3
3·3 = 1 + 1·8

FIGURE 37
The Whitehead link is strongly invertible

In general we have the following

THEOREM 7. *Let* $L = N(T)$ *be an oriented rational link with associated tangle fraction* $F(T) = p/q$ *of parity* e/o, *with* p *and* q *relatively prime and* $|p| > |q|$. *Then* L *is strongly invertible if and only if* $q^2 = 1 + up$ *with* u *an odd integer. It follows that strongly invertible links are all numerators of rational tangles of the form* $[[a_1], [a_2], \ldots, [a_k], [\alpha], [a_k], \ldots, [a_2], [a_1]]$ *for any integers* $a_1, \ldots, a_k, \alpha$.

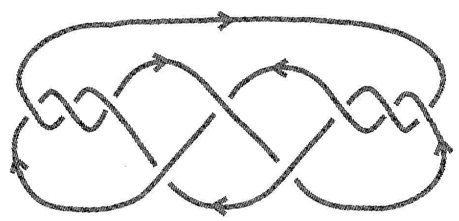
Proof. In T the upper two end arcs close to form one component of L and the lower two end arcs close to form the other component of L. Let T' denote the tangle obtained from the oriented tangle T by reversing the orientation of the component containing the lower two arcs and let $N(T') = L'$. (If T'' denotes the tangle obtained from the oriented tangle T by reversing the orientation of the component containing the upper two arcs we have seen that by a vertical 180° rotation the link $N(T')$ is isotopic to the link $N(T'')$. So, for proving Theorem 7 it suffices to consider only the case above.)

Note that T and T' are incompatible. Thus to apply Theorem 3 we need to perform a bottom twist on T'. Since T and T' have the same fraction p/q, after applying the twist we need to compare the fractions p/q and $p/(p+q)$. Since q is not congruent to $(p+q)$ modulo $2p$, we need to determine when $q(p+q)$ is congruent to 1 modulo $2p$. This will happen exactly when $qp + q^2 = 1 + 2Kp$ for some integer K. The last equation is the same as saying that $q^2 = 1 + up$ with $u = 2K - q$ odd, since q is odd. Now it follows from the Palindrome Theorem for continued fractions that $q^2 = 1 + up$ with u odd and p even if and only if the fraction p/q with $|p| > |q|$ has a palindromic continued fraction expansion with an odd number of terms (the proof is the same in form as the corresponding argument given in the proof of Theorem 5). That is, it has a continued fraction in the form

$$[a_1, a_2, \ldots, a_n, \alpha, a_n, a_{n-1}, \ldots, a_2, a_1].$$

It is then easy to see that the corresponding rational link is ambient isotopic to itself through a vertical 180° rotation. Hence it is strongly invertible. It follows from this that all strongly invertible rational links are ambient isotopic to themselves through a 180° rotation just as in the example of the Whitehead link given above. This completes the proof of the Theorem. □

REMARK 7. Excluding the possibility $T = [\infty]$, as $F(T) = 1/0$ does not have the parity e/o, we may assume $q \neq 0$. And since q is odd (in order that the rational tangle has two components), the integer $u = 2K - q$ in the equation $q^2 = 1 + up$ cannot be zero. It follows then that the links of the type $N([2n])$, for $n \in \mathbf{Z}$, $n \neq 0$, with tangle fraction $2n/1$ are not invertible (recall the example in Figure 30). Note that, for $n = 0$ we have $T = [0]$ and $F(T) = 0/1$, and in this case Theorem 7 is confirmed, since $1^2 = 1 + u0$, for any u odd. See Figure 38 for another example of a strongly invertible link. In this case the link is $L = N([[3],[1],[1],[1],[3]])$ with $F(L) = 40/11$. Note that $11^2 = 1 + 3 \cdot 40$, fitting the conclusion of Theorem 7.



L = N([[3], [1], [1], [1], [3]])

FIGURE 38

An example of a strongly invertible link

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