6. The oriented case

$$\chi * \chi = {>}{<}$$
$$\chi * \asymp = \asymp$$
$$\asymp * \asymp = \delta \asymp$$
$$\chi * {>}{<} = \chi$$
$$\asymp * {>}{<} = \asymp$$
$${>}{<} * {>}{<} = {>}{<}$$

The symbol δ stands for the value of a loop formed. Now any rational tangle can be built from [0] or [∞] by successive addition or multiplication with [± 1]. Thus, from the point of view of connectivity, it suffices to show that $[T]+[\pm 1]$ and $[T]*[\pm 1]$ satisfy the theorem whenever $[T]$ satisfies the theorem. This is checked by comparing the connectivity identities above with the parity of the fractions. For example, in the case

$$\chi + \chi = \asymp \quad \text{we have } o/o + o/o = e/o$$

exactly in accordance with the connectivity identity. The other cases correspond as well, and this proves the theorem by induction. □

COROLLARY 1. *For a rational tangle T the link $N(T)$ has two components if and only if T has fraction $F(T)$ of parity e/o.*

Proof. By the Theorem we have $F(T)$ has parity e/o if and only if T has connectivity type $\asymp$. It follows at once that $N(T)$ has two components. □

Another useful fact is that the components of a rational link are individually unknotted embeddings in three dimensional space. To see this, observe that upon removing one strand of a rational tangle, the other strand is an unknotted arc.

6. THE ORIENTED CASE

Oriented rational knots and links are numerator (and denominator) closures of oriented rational tangles. Rational tangles are oriented by choosing an orientation for each strand of the tangle. Two oriented rational tangles are *isotopic* if they are isotopic as unoriented tangles via an isotopy that carries the orientation of one tangle to the orientation of the other. Since the end arcs of a tangle are fixed during a tangle isotopy, this means that isotopic tangles must have identical orientations at their end arcs. Thus, *two oriented tangles are isotopic if and only if they are isotopic as unoriented tangles and they have identical orientations at their end arcs.* It follows that a given unoriented rational tangle can always yield non-isotopic oriented rational tangles, for different choices of orientation of one or both strands.

In order to compare oriented rational knots via rational tangles we are only interested in orientations that yield consistently oriented knots upon taking the numerator closure. This means that the two top end arcs have to be oriented one inward and the other outward. Same for the two bottom end arcs.

Reversing the orientation of one strand of an oriented rational tangle that gives rise to a two-component link will usually yield non-isotopic oriented rational links. Figure 30 illustrates an example of non-isotopic oriented rational links, which are isotopic as unoriented links. But reversing a single strand may also yield isotopic oriented rational links. This will be the subject of the next section.

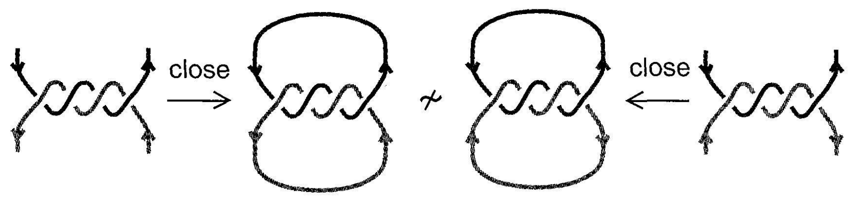


FIGURE 30
Non-isotopic oriented rational links

An oriented knot or link is said to be *invertible* if it is oriented isotopic to its inverse, i.e. the link obtained from it by reversing the orientation of each component. We can obtain the inverse of a rational link by reversing the orientation of both strands of the oriented rational tangle of which it is the numerator. It is easy to see that any rational knot or link is invertible. See the example on the right-hand side of Figure 31.

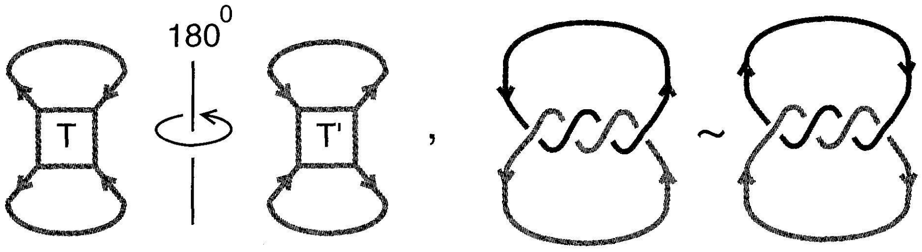


FIGURE 31
Isotopic oriented rational knots and links

LEMMA 2. *Rational knots and links are invertible.*

Proof. Let $K = N(T)$ be an oriented rational knot or link with T an oriented rational tangle. We do a vertical $180°$-rotation in space, as the left-hand side of Figure 31 demonstrates. This rotation is a vertical flip for the rational tangle T. Let T' denote the result of the vertical flip of the tangle T. The resulting oriented knot $K' = N(T')$ is oriented isotopic to K, while the orientation of T' is the opposite of that of T on both strands, and thus on all end arcs. But as we have already noted T is isotopic to its vertical flip as unoriented tangles, thus they will have the same fraction. It follows that T' can be isotoped to T through an (unoriented) isotopy that leaves the external strands fixed. Therefore, the result of the vertical $180°$-rotation is the tangle T but with all orientations reversed. Thus K' is the inverse of K, and from the above K is oriented isotopic to its inverse. □

Using this observation we conclude that, as far as the study of oriented rational knots is concerned, *all oriented rational tangles may be assumed to have the same orientation for their two upper end arcs.* Indeed, if the orientations of the two upper end arcs are opposite of the fixed ones we do a vertical flip to obtain the orientation pattern that agrees with our convention. We fix this orientation to be downward for the NW end arc and upward for the NE end arc, as in the examples of Figure 30 and as illustrated in Figure 32.

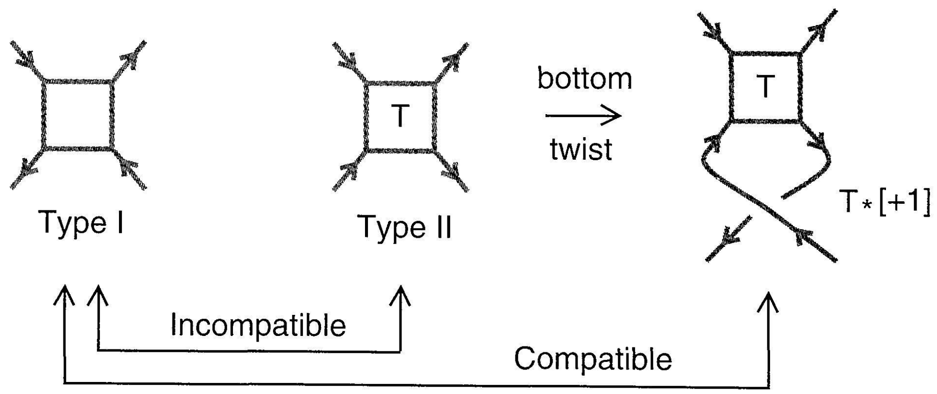


FIGURE 32
Compatible and incompatible orientations

Thus we may reduce our analysis to two basic types of orientation for the four end arcs of a rational tangle. We shall call an oriented rational tangle *of type I* if the SW arc is oriented downward and the SE arc is oriented upward, and *of type II* if the SW arc is oriented upward and the SE arc is oriented downward, see Figure 32. From the above remarks any tangle is of type I or type II. Two tangles are said to be *compatible* it they are both of type I or both of type II and *incompatible* if they are of different types.

In order to classify oriented rational knots, seen as numerator closures of oriented rational tangles, we will always compare compatible rational tangles.

While the connectivity type of unoriented rational tangles may be [0], [∞] or [1], note that an oriented rational tangle of type I will have connectivity type [0] or [∞] and an oriented rational tangle of type II will have connectivity type [0] or [1].

BOTTOM TWIST BASICS. If two oriented tangles are incompatible, adding a single half twist at the bottom of one of them yields a new pair of compatible tangles, as Figure 32 illustrates. Note also that adding such a twist, although it changes the tangle, does not change the isotopy type of the numerator closure. Thus, up to bottom twists, we are always able to compare oriented rational tangles of the same orientation type. Further, note that if we add a positive bottom twist to an oriented rational tangle T with fraction $F(T) = p/q$ we obtain the incompatible tangle $T' = T * [+1]$ with fraction $F(T') = 1/(1 + 1/F(T)) = p/(p + q)$. Similarly, if we add a negative twist we obtain the incompatible tangle $T'' = T * [-1]$ with fraction $F(T'') = 1/(-1 + 1/F(T)) = p/(-p + q)$. It is worth noting here that the tangles T' and T'' are compatible and $p+q \equiv (-p+q) \mod 2p$, confirming the Oriented Schubert Theorem.

Schubert [31] proved his version of Theorem 3 by using the 2-bridge representation of rational knots and links. We give a tangle-theoretic proof of Schubert's Oriented Theorem, based upon the combinatorics of the unoriented case and then analyzing how orientations and fractions are related.

In our statement of Theorem 3 in the introduction we restricted the denominators of the fractions to be odd. This is a restriction made for the purpose of comparison of tangles. There is no loss of generality, as will be seen when we analyze the palindrome case in the proof at the end of this section. What happens is this: In the case of p odd and only one of q and q' even, one finds that the corresponding tangles are incompatible. We can then compare them by adding a bottom twist to one of the tangles. Adding this twist, the even denominator is replaced by an odd denominator. In the case where p is odd and both q and q' are even, one finds that the corresponding tangles are compatible. In this case, we add a twist at the bottom of each tangle to preserve the hypothesis that both denominators are odd. This extra twisting yields compatible tangles and a successful comparison.

The strategy of our proof is as follows. Consider an oriented rational knot or link diagram K given in standard form as $N(T)$, where T is a rational tangle in continued fraction form. Our previous analysis tells us that, up to bottom twists, any other rational tangle that closes to this knot is available as a cut on the given diagram. If two rational tangles close to give K as an unoriented rational knot or link, then there are orientations on these tangles, induced from K, so that the oriented tangles close to give K as an oriented knot or link. Two tangles so produced may or may not be compatible. However, adding a bottom twist to one of two incompatible tangles results in two compatible tangles. *It is this possible twist difference that gives rise to the change from modulus p in the unoriented case to the modulus $2p$ in the oriented case.*

We now analyze when, comparing with the original standard cut, another cut produces a compatible or incompatible tangle. See Figure 34 for an example illustrating the compatibility of orientations in the case of the palindrome cut. Note that reducing all possible bottom twists implies $|p| > |q|$ for both tangles, if the two reduced tangles that we compare each time are compatible, or for only one, if they are incompatible. Recall Figure 12 and the related analysis for the basic arithmetic of the bottom twists.

EVEN BOTTOM TWISTS. The simplest instance of the classification of oriented rational knots is adding an *even number of twists* at the bottom of an oriented rational tangle T. We then obtain a compatible tangle $T * 1/[2n]$, and $N(T * 1/[2n]) \sim N(T)$. If now $F(T) = p/q$, then $F(T * 1/[2n]) = F(1/([2n] + 1/T)) = 1/(2n + 1/F(T)) = p/(2np + q)$. Hence, if we set $2np + q = q'$ we have

$$q \equiv q' \mod 2p,$$

just as Theorem 3 predicts.

We then have to compare the special cut and the palindrome cut with the standard cut. Here also, the special cut is the easier to see whilst the palindrome cut requires a more sophisticated analysis. Figure 17 explained how to obtain the unoriented tangle of the special cut. Moreover, by Remark 2, adding a bottom twist to the tangle of the special cut yields a tangle isotopic to the tangle of the standard cut.

Figure 33 demonstrates that the special cut yields oriented incompatible tangles. More precisely, in the case of the special cut we are presented with the general fact that for any tangle R, $N([+1] + R)$ and $N([-1] - 1/R)$

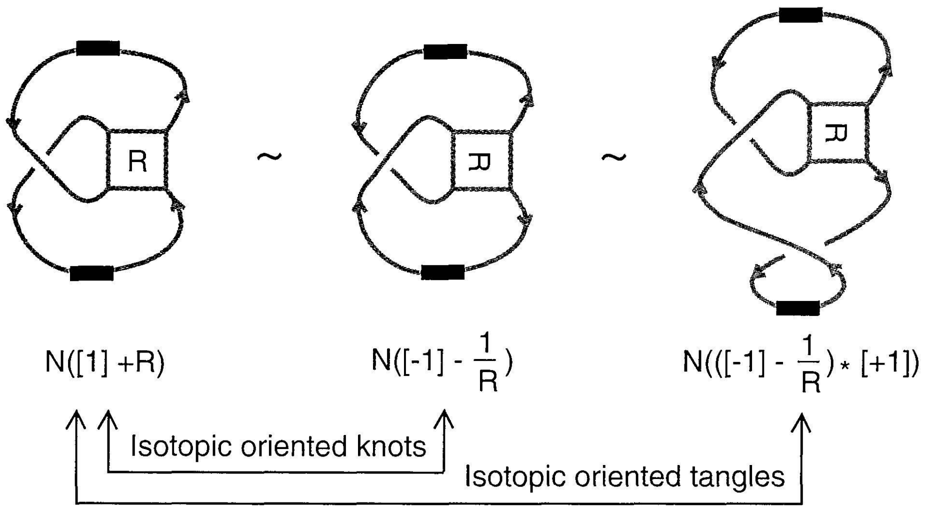


FIGURE 33
The oriented special cut yields incompatible tangles

are unoriented isotopic. With orientations coming from the cut we find that $S = [+1] + R$ and $S' = [-1] - 1/R$ are incompatible. Adding a bottom twist yields oriented compatible tangles, which from the above are isotopic. So, there is nothing to check and the Oriented Schubert Theorem is verified in the strongest possible way for the oriented special cut.

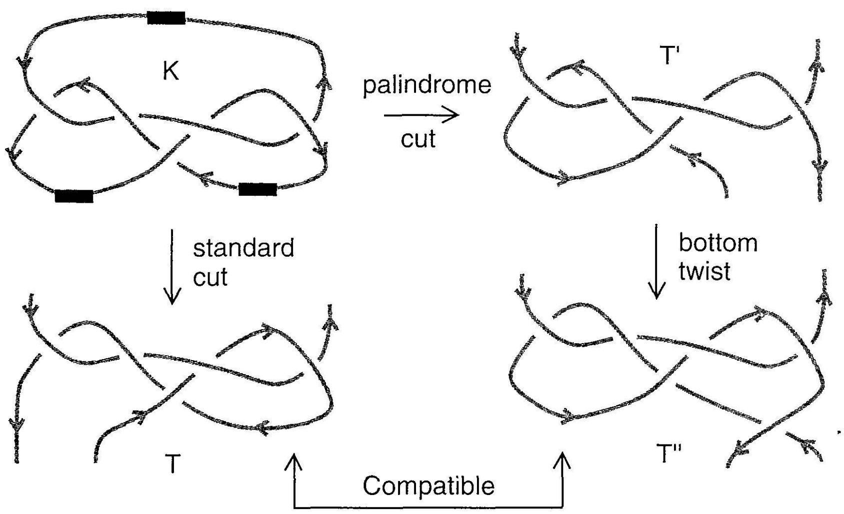


FIGURE 34
Oriented standard cut and palindrome cut

We are left to examine the case of the palindrome cut. In order to analyze this case, we must understand when the standard cut and the palindrome cut are compatible or incompatible. Then we must compare their respective fractions. Figure 34 illustrates how compatibility is obtained by using a bottom twist, in the case of a palindrome cut. In this example we illustrate the standard and palindrome cuts on the oriented rational knot $K = N(T) = N(T')$ where $T = [[2],[1],[2]]$ and T' its palindrome. As we can see, the two cuts place incompatible orientations on the tangles T and T'. Adding a twist at the bottom of T' produces a tangle $T'' = T' * [-1]$ that is compatible with T. Now we compute $F(T) = F(T') = 8/3$ and $F(T'') = F(T' * [-1]) = 8/-5$ and we notice that $3 \cdot (-5) \equiv 1 \mod 16$, as Theorem 3 predicts.

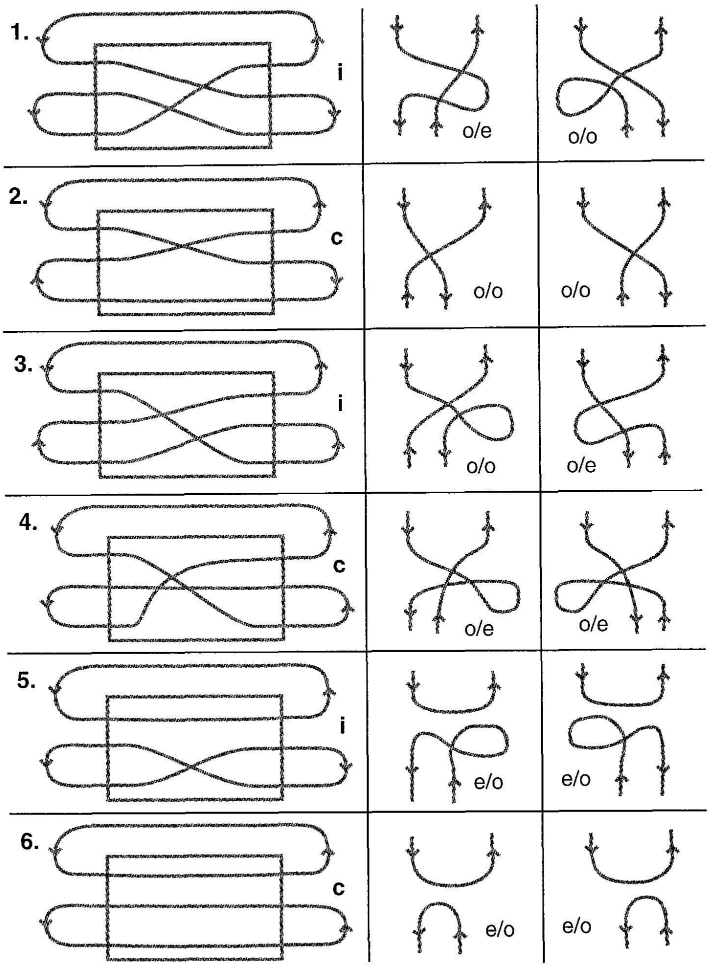


FIGURE 35
The six connection structures, compatibility and parity of the palindrome cut

The study of the compatibility or not of the palindrome cut involves a deeper analysis along the lines of Theorem 6. With the issues of connectivity in place we can begin to analyze the different connectivities and parities in the standard and palindrome cuts on a rational knot or link in standard 3-strand-braid representation. See Figure 35. In this figure we have enumerated the six connection structures for a 3-strand braid (corresponding to the six permutations of three points) with plat closures (of the braid augmented by an extra strand) corresponding to oriented rational knots and links. These closed connection patterns shall be called *connectivity charts*. We then show corresponding to each connectivity chart the related standard and palindrome cuts and the connectivity and parity of the corresponding tangles. Compatibility or incompatibility of these tangles, specified by an 'i' or 'c', can be read from the oriented diagrams in the figure.

Proof of the palindrome cut. It suffices to verify the Theorem in all cases of the comparison of standard and palindrome cuts on a rational knot K in continued fraction form. We can assume that $K = N([[a_1],\dots,[a_n]])$ with n odd. Then the tangle $T = [[a_1],\dots,[a_n]]$ is, by construction, the standard cut on K. We know that the matrix product

$$M = M(a_1)M(a_2)\cdots M(a_n) = \begin{pmatrix} p & q' \\ q & u \end{pmatrix}$$

encodes the fractions of T and its palindrome $T' = [[a_n],\dots,[a_1]]$, with $F(T) = p/q$ and $F(T') = p/q'$. Note that, since $Det(M) = -1$, we have the formula

$$q\,q' = 1 + up$$

relating the denominators of these fractions.

CASE 1. p ODD, PART A:

If only one of q or q' is even (parts 1 and 3 of Figure 35), then the fact that $q\,q' = 1 + up$ implies the parity equation $e = 1 + uo$, hence *u is odd.* Now refer to Figure 35 and note that the standard and palindrome cuts are incompatible in both cases 1 and 3. (The cases are $\{o/e, o/o\}$ and $\{o/o, o/e\}$.) In order to obtain compatibility, add a bottom twist to the cut with even denominator. Without loss of generality, we may assume that q' is even, so that we will compare p/q and $p/(p+q')$. Note that

$$q(p+q') = qp + q\,q' = qp + 1 + up = 1 + (u+q)p.$$

Since u is odd and q is odd, it follows that $(u+q)$ is even. Hence, $q(p+q') \equiv 1$ mod $2p$, proving Theorem 3 in this case.

CASE 1. p ODD, PART B:

Now suppose that *both q and q' are even.* We are in part 4 of Figure 35 and the two cuts are compatible. Therefore we apply a bottom twist to each cut giving the fractions $p/(p+q)$ and $p/(p+q')$ for comparison. Note that

$$(p+q)(p+q') = p^2 + qp + q'p + q\,q' = 1 + (p+q+q'+u)p$$

and we have the parity equation

$$p + q + q' + u = o + e + e + o = e .$$

Hence $(p+q)(p+q') \equiv 1 \mod 2p$ verifying the Theorem in this case.

CASE 1. p ODD, PART C:

Finally (for Case 1) suppose that q *and* q' *are both odd.* Then the parity equation corresponding to $qq' = 1 + up$ is

$$o = 1 + uo .$$

Hence u is even so that $qq' \equiv 1 \mod 2p$. We are in part 2 of Figure 35, and the standard and palindrome cuts are compatible. This is in accord with the congruence above, hence the Theorem is verified in this case.

CASE 2. p EVEN:

Now we assume that p is even. This corresponds to parts 5 and 6 in Figure 35 (two components). In part 5 the cuts are compatible, while in part 6 the cuts are incompatible. In either case, both q and q' are odd so that the fractions p/q and p/q' both have the parity e/o. The equation $qq' = 1 + up$ has corresponding parity equation $o = 1 + ue$, and u can be either even or odd. In order to accomplish the proof of Case 2 we will show that

1. u is even if and only if the standard and palindrome cuts are compatible.
2. u is odd if and only if the standard and palindrome cuts are incompatible.

We prove these statements by induction on the number of terms in the continued fraction $[a_1, \ldots, a_n]$. The induction step consists in adding two more terms to the continued fraction (thereby maintaining an odd number of terms). That is, we shall examine a continued fraction in the form $T_{n+2} = [a_1, \ldots, a_{n+2}]$ that is *given* to be in cases 5 or 6 of Figure 35. See Figure 36. In Figure 36 the numbers that label the diagrams refer to the cases in Figure 35. We consider the structure of the "predecessor" $T_n = [a_1, \ldots, a_n]$ of T_{n+2} which may be in the form 5 or 6, as shown in Figure 36 (in which case we can apply the induction hypothesis) or it may be in one of the other four cases shown in Figure 36.

In Figure 36 we have shown the connectivity patterns that result in T_{n+2} landing in cases 5 or 6. In this figure the rectangular boxes indicate the internal connectivity of T_n, and we have separated these specific cases into three types

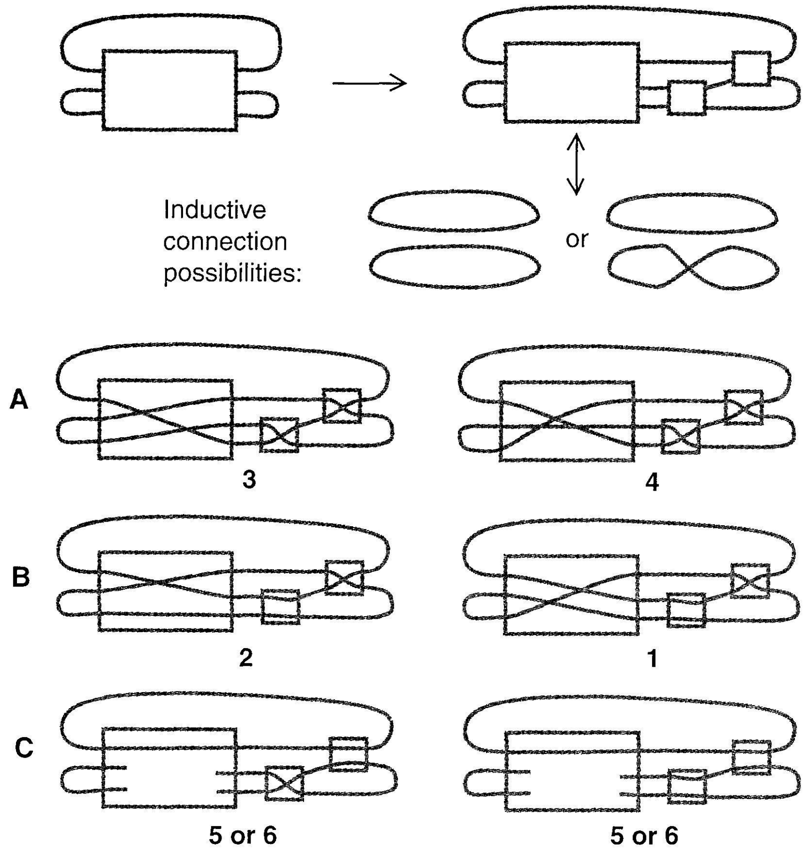


FIGURE 36
Inductive connections

labeled A, B and C (not to be confused with subcases of this proof). In this figure each case is labeled with the type of the predecessor. Thus in the A row one sees the labels 3 and 4 because the boxed patterns are respectively of types 3 and 4. In rows A and B the left hand entries are of type 6 after the addition of the two new terms, and the right hand entries are of type 5. We then check each of these cases to see that the induced value of u in T_{n+2} has the right parity. The calculations can be done by multiplication of generating matrices for continued fractions just using the parity algebra. For example, in Case A of Figure 36 we add two new odd parity terms to T_n in order to obtain T_{n+2}. Thus we multiply the parity matrix for T_n by the product

$$\begin{pmatrix} o & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} o & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e & o \\ o & 1 \end{pmatrix}$$

in order to obtain the parity matrix for T_{n+2}.

In particular, if T_n is in case 3 of Figure 35, then it has fraction parities o/o and o/e and hence parity matrix $\begin{pmatrix} o & e \\ o & o \end{pmatrix}$. Multiplying this by $\begin{pmatrix} e & o \\ o & 1 \end{pmatrix}$, we obtain

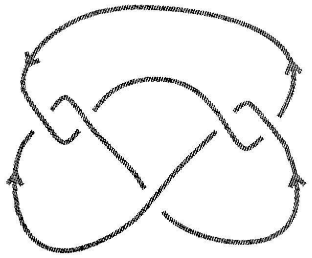
$$\begin{pmatrix} o & e \\ o & o \end{pmatrix} \begin{pmatrix} e & o \\ o & 1 \end{pmatrix} = \begin{pmatrix} e & o \\ o & e \end{pmatrix}.$$

Thus the new u for T_{n+2} is even. Since the connectivity diagram for T_{n+2} in this case, as shown in Figure 36, has compatible standard and palindrome cuts, this result for the parity of u is one step in the verification of the induction hypothesis. Each of the six cases is handled in this same way. We omit the remaining details and assert that the values of u obtained in each case are correct with respect to the connection structure. This completes the proof of Case 2.

Since Cases 1 and 2 encompass all the different possibilities for the standard and palindrome cuts, this completes the proof of the Oriented Schubert Theorem. □

7. Strongly Invertible Links

An oriented knot or link is invertible if it is oriented isotopic to the link obtained from it by reversing the orientation of each component. We have seen (Lemma 2) that rational knots and links are invertible. A link L of two components is said to be *strongly invertible* if L is ambient isotopic to itself with the orientation of only one component reversed. In Figure 37 we illustrate the link $L = N([[2],[1],[2]])$. This is a strongly invertible link as is apparent by a 180° vertical rotation. This link is well-known as the Whitehead link, a link with linking number zero. Note that since $[[2],[1],[2]]$ has fraction equal to $2 + 1/(1 + 1/2) = 8/3$ this link is non-trivial via the classification of rational knots and links. Note also that $3 \cdot 3 = 1 + 1 \cdot 8$.



N([[2], [1], [2]]) = W
the Whitehead Link
F(W) = 2+1/(1+1/2) = 8/3
3·3 = 1 + 1·8

FIGURE 37
The Whitehead link is strongly invertible