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ON THE CLASSIFICATION OF RATIONAL KNOTS

by Louis H. KAUFFMAN and Sofia LAMBROPOULOU

ABSTRACT. In this paper we give combinatorial proofs of the classification of unoriented and oriented rational knots based on the now known classification of alternating knots and the calculus of continued fractions. We also characterize the class of strongly invertible rational links. Rational links are of fundamental importance in the study of DNA recombination.

1. INTRODUCTION

Rational knots and links comprise the simplest class of links. The first twenty five knots, except for 8_5, are rational. Furthermore all knots and links up to ten crossings are either rational or are obtained by inserting rational tangles into a small number of planar graphs, see [6]. Rational links are alternating with one or two unknotted components, and they are also known in the literature as Viergeflechte, four-plats or 2-bridge knots depending on their geometric representation. More precisely, rational knots can be represented as:

• plat closures of four-strand braids (Viergeflechte [1], four-plats). These are knot diagrams with two local maxima and two local minima.

• 2-bridge knots. A 2-bridge knot is a knot that has a diagram in which there are two distinct arcs, each overpassing a consecutive sequence of crossings, and every crossing in the diagram is in one of these sequences. The two arcs are called the bridges of the diagram (compare with [5], p. 23).

• numerator or denominator closures of rational tangles (see Figures 1, 5). A rational tangle is the result of consecutive twists on neighboring endpoints of two trivial arcs. For examples see Figure 1 and Figure 3.

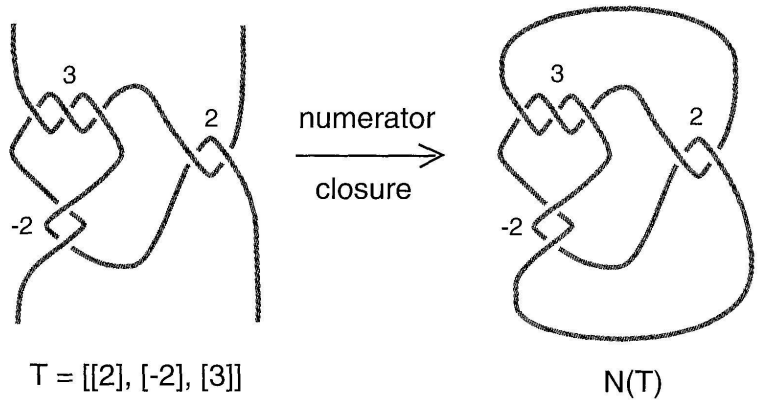


FIGURE 1
A rational tangle and a rational knot

All three representations are equivalent. The equivalence between the first and the third is easy to see by planar isotopies. For the equivalence between the first and the second representation see for example [5], pp. 23, 24. In this paper we consider rational knots as obtained by taking numerator or denominator closures of rational tangles (see Figure 5).

The notion of a tangle was introduced in 1967 by Conway [6] in his work on enumerating and classifying knots and links, and he defined the rational knots as numerator or denominator closures of the rational tangles. (It is worth noting here that Figure 2 in [1] illustrates a rational tangle, but no special importance is given to this object. It is obtained from a four-strand braid by plat-closing only the top four ends.) Conway [6] also defined *the fraction* of a rational tangle to be a rational number or ∞. He observed that this number for a rational tangle equals a continued fraction expression with all numerators equal to one and all denominators of the same sign, that can be read from a tangle diagram in alternating standard form. Rational tangles are classified by their fractions by means of the following theorem.

THEOREM 1 (Conway, 1975). *Two rational tangles are isotopic if and only if they have the same fraction.*

Proofs of Theorem 1 are given in [21], [5] p. 196, [13] and [15]. The first two proofs invoked the classification of rational knots and the theory of branched covering spaces. The 2-fold branched covering spaces of S^3 along the rational links give rise to the lens spaces $L(p, q)$. See [33] for a pioneering treatment of branched coverings. The proof in [13] is the first combinatorial proof of this theorem. The proofs in [21], [5] and [13] use definitions different

from the above for the fraction of a rational tangle. In [15] a new combinatorial proof of Theorem 1 is given using the solution of the Tait Conjecture for alternating knots [42], [20] adapted for tangles. A second combinatorial proof is given in [15] using coloring for defining the tangle fraction.

Throughout the paper by the term 'knots' we will refer to both knots and links, and whenever we really mean 'knot' we shall emphasize it. More than one rational tangle can yield the same or isotopic rational knots and the equivalence relation between the rational tangles is mapped into an arithmetic equivalence of their corresponding fractions. Indeed we have the following

THEOREM 2 (Schubert, 1956). *Suppose that rational tangles with fractions* $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *are given (p and q are relatively prime; similarly for* p' *and* q'*). If* $K(\frac{p}{q})$ *and* $K(\frac{p'}{q'})$ *denote the corresponding rational knots obtained by taking numerator closures of these tangles, then* $K(\frac{p}{q})$ *and* $K(\frac{p'}{q'})$ *are isotopic if and only if*

1. $p = p'$ *and*
2. *either* $q \equiv q' \mod p$ *or* $qq' \equiv 1 \mod p$.

Schubert [31] originally stated the classification of rational knots and links by representing them as 2-bridge links. Theorem 2 has hitherto been proved by taking the 2-fold branched covering spaces of S^3 along 2-bridge links, showing that these correspond bijectively to oriented diffeomorphism classes of lens spaces, and invoking the classification of lens spaces [28]. Another proof using covering spaces has been given by Burde in [4]. See also the excellent notes on the subject by Siebenmann [35]. The above statement of Schubert's theorem is a formulation of the Theorem in the language of Conway's tangles.

Using his methods for the unoriented case, Schubert also extended the classification of rational knots and links to the case of oriented rational knots and links described as 2-bridge links. Here is our formulation of the Oriented Schubert Theorem written in the language of Conway's tangles.

THEOREM 3 (Schubert, 1956). *Suppose that orientation-compatible rational tangles with fractions* $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *are given with q and* q' *odd (p and q are relatively prime; similarly for* p' *and* q'*). If* $K(\frac{p}{q})$ *and* $K(\frac{p'}{q'})$ *denote the corresponding rational knots obtained by taking numerator closures of these tangles, then* $K(\frac{p}{q})$ *and* $K(\frac{p'}{q'})$ *are isotopic if and only if*

1. $p = p'$ *and*
2. *either* $q \equiv q' \mod 2p$ *or* $qq' \equiv 1 \mod 2p$.

Theorems 2 and 3 could have been stated equivalently using the denominator closures of rational tangles. Then the arithmetic equivalences of the tangle fractions related to isotopic knots would be the same as in Theorems 2 and 3, but with the roles of numerators and denominators exchanged.

This paper gives the first combinatorial proofs of Theorems 2 and 3 using tangle theory. Our proof of Theorem 2 uses the results and the techniques developed in [15], while the proof of Theorem 3 is based on that of Theorem 2. We have located the essential points in the proof of the classification of rational knots in the question: *Which rational tangles will close to form a specific knot or link diagram?* By looking at the Theorems in this way, we obtain a path to the results that can be understood without extensive background in three-dimensional topology. In the course of these proofs we see connections between the elementary number theory of fractions and continued fractions, and the topology of knots and links. In order to compose these proofs we use the fact that rational knots are alternating (which follows from the fact that rational tangles are alternating, and for which we believe we found the simplest possible proof, see [15], Proposition 2). We then rely on the *Tait Conjecture* [42] concerning the classification of alternating knots, which states the following:

Two alternating knots are isotopic if and only if any two corresponding reduced diagrams on S^2 are related by a finite sequence of flypes (see Figure 6).

A diagram is said to be *reduced* if at every crossing the four local regions indicated at the crossing are actually parts of four distinct global regions in the diagram (see [19], p. 42). It is not hard to see that any knot or link has reduced diagrams that represent its isotopy class. The conjecture was posed by P. G. Tait [42] in 1877 and was proved by W. Menasco and M. Thistlethwaite, [20] in 1993. Tait did not actually phrase this statement as a conjecture. It was a working hypothesis for his efforts in classifying knots.

Our proof of the Schubert Theorem is elementary upon assuming the Tait Conjecture, but this is easily stated and understood. This paper will be of interest to mathematicians and biologists.

The paper is organized as follows. In Section 2 we give the general set up for rational tangles, their isotopies and operations, as well as their association to a continued fraction isotopy invariant. In this section we also recall the basic theory and a canonical form of continued fractions. In Section 3 we prove Theorem 2 about the classification of unoriented rational knots by means of a direct combinatorial and arithmetical analysis of rational knot diagrams,

using the classification of rational tangles and the Tait Conjecture. In Section 4 we discuss chirality of knots and give a classification of the achiral rational knots and links as numerator closures of even palindromic rational tangles in continued fraction form (Theorem 5). In Section 5 we discuss the connectivity patterns of the four end arcs of rational tangles and we relate connectivity to the parity of the fraction of a rational tangle (Theorem 6). In Section 6 we give our interpretation of the statement of Theorem 3 and we prove the classification of oriented rational knots, using the methods we developed in the unoriented case and examining the connectivity patterns of oriented rational knots. In Section 6 it is pointed out that all oriented rational knots and links are invertible (reverse the orientation of both components). In Section 7 we give a classification of the strongly invertible rational links (reverse the orientation of one component) as closures of odd palindromic oriented rational tangles in continued fraction form (Theorem 7).

Here is a short history of the theory of rational knots. As explained in [14], rational knots and links were first considered by O. Simony in 1882, [36, 37, 38, 39], taking twistings and knottings of a band. Simony [37] was the first one to relate knots to continued fractions. After about sixty years Tietze wrote a series of papers [43, 44, 45, 46] with reference to Simony's work. Reidemeister [27] in 1929 calculated the knot group of a special class of four-plats (Viergeflechte), but four-plats were really studied by Goeritz [12] and by Bankwitz and Schumann [1] in 1934. In [12] and [1] proofs are given independently and with different techniques that rational knots have 3-strand-braid representations, in the sense that the first strand of the four-strand braids can be free of crossings, and that they are alternating. (See Figure 20 for an example and Figure 26 for an abstract 3-strand-braid representation.) The proof of the latter in [1] can be easily applied on the corresponding rational tangles in standard form. (See Figure 1 for an example and Figure 8 for abstract representations.)

In 1954 Schubert [30] introduced the bridge representation of knots. He then showed that the four-plats are exactly the knots that can be represented by diagrams with two bridges and consequently he classified rational knots by finding canonical forms via representing them as 2-bridge knots, see [31]. His proof was based on Seifert's observation that the 2-fold branched coverings of 2-bridge knots [33] give rise to lens spaces and on the classification of lens spaces by Reidemeister [28] using Reidemeister torsion and following the lead of [32] (and later by Brody [3] using the knot theory of the lens space). See also [25]. Rational knots and rational tangles figure prominently in the applications of knot theory to the topology of DNA, see [40]. Treatments of

various aspects of rational knots and rational tangles can be found in many places in the literature, see for example [6], [35], [29], [5], [2], [22], [16], [19].

2. Rational tangles and their invariant fractions

In this section we recall from [15] the facts that we need about rational tangles, continued fractions and the classification of rational tangles. We intend the paper to be as self-contained as possible.

A 2-*tangle* is a proper embedding of two unoriented arcs and a finite number of circles in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3. A *rational tangle* is a proper embedding of two unoriented arcs α_1, α_2 in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3, and such that there exists a homeomorphism of pairs:

$$\overline{h}\colon (B^3, \alpha_1, \alpha_2) \to (D^2 \times I,\ \{x, y\} \times I) \quad \text{(a trivial tangle)}.$$

This is equivalent to saying that rational tangles have specific representatives obtained by applying a finite number of consecutive twists of neighboring endpoints starting from two unknotted and unlinked arcs. Such a pair of arcs comprise the [0] or [∞] tangles, depending on their position in the plane, see illustrations in Figure 2.

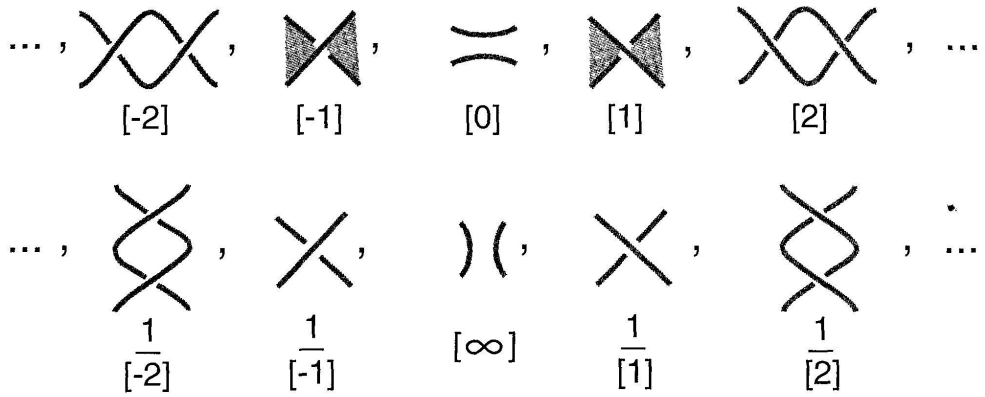


FIGURE 2
The elementary rational tangles and the types of crossings

We shall use this characterizing property of a rational tangle as our definition, and we shall then say that the rational tangle is in *twist form*. See Figure 3 for an example.

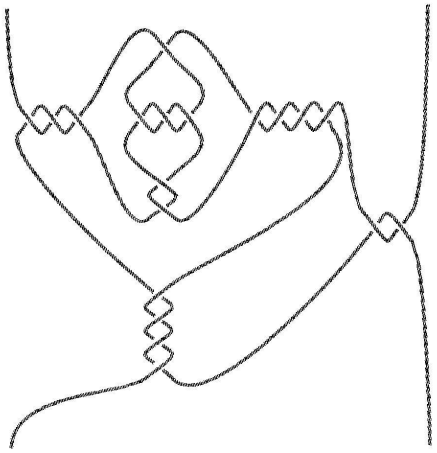


FIGURE 3
A rational tangle in twist form

To see the equivalence of the above definitions, let S^2 denote the two-dimensional sphere, which is the boundary of the 3-ball B^3 and let p denote four specified points in S^2. Let further $h\colon (S^2,p) \to (S^2,p)$ be a self-homeomorphism of S^2 with the four points. This extends to a self-homeomorphism $\overline{h}$ of the 3-ball B^3 (see [29], page 10). Further, let a denote the two straight arcs $\{x,y\}\times I$ joining pairs of the four points in the boundary of B^3. Consider now $\overline{h}(a)$. We call this the tangle induced by h. We note that up to isotopy (see definition below) h is a composition of braidings of pairs of points in S^2 (see [24], pages 61 to 65). Each such braiding induces a twist in the corresponding tangle. So, if h is a composition of braidings of pairs of points, then the extension $\overline{h}$ is a composition of twists of neighboring end arcs. Thus $\overline{h}(a)$ is a rational tangle and every rational tangle can be obtained this way.

A *tangle diagram* is a regular projection of the tangle on a meridian disc. Throughout the paper by 'tangle' we will mean 'regular tangle diagram'. The type of crossings of knots and 2-tangles follow the checkerboard rule: shade the regions of the tangle (knot) in two colors, starting from the left (outside) to the right (inside) with grey, and so that adjacent regions have different colors. Crossings in the tangle are said to be of *positive type* if they are arranged with respect to the shading as exemplified in Figure 2 by the tangle $[+1]$, i.e. they have the region on the right shaded as one walks towards the crossing along the over-arc. Crossings of the reverse type are said to be of *negative type* and they are exemplified in Figure 2 by the tangle $[-1]$. The reader should note that our crossing type conventions are the opposite of those of Conway in [6] and of those of Kawauchi in [16]. Our conventions agree with those of Ernst and Sumners [10], [40] which in turn follow the standard conventions of biologists.

We are interested in tangles up to isotopy. Two rational tangles, T, S, in B^3 are *isotopic*, denoted by $T \sim S$, if and only if any two diagrams of them have identical configurations of their four endpoints on the boundary of the projection disc, and they differ by a finite sequence of the well-known Reidemeister moves [27], which take place in the interior of the disc. Of course, each twisting operation used in the definition of a rational tangle changes the isotopy class of the tangle to which it is applied.

2-TANGLE OPERATIONS. The symmetry of the four endpoints of 2-tangles allows for the following well-defined (up to isotopy) operations in the class of 2-tangles, as described in Figure 4. We have *the sum* of two 2-tangles, denoted by '$+$' and *the product* of two 2-tangles, denoted by '$*$'. This product '$*$' is not to be confused with Conway's product '$\cdot$' in [6].

In view of these operations we can say that a rational tangle is created inductively by consecutive additions of the tangles $[\pm 1]$ on the right or on the left and multiplications by the tangles $[\pm 1]$ at the bottom or at the top, starting from the tangles $[0]$ or $[\infty]$. And since, when we start creating a rational tangle, the very first crossing can be equally seen as a horizontal or as a vertical one, we may always assume that we start twisting from the tangle $[0]$. Addition and multiplication of tangles are not commutative. Also, they do not preserve the class of rational tangles. The sum (product) of two rational tangles is rational if and only if one of the two consists in a number of horizontal (vertical) twists.

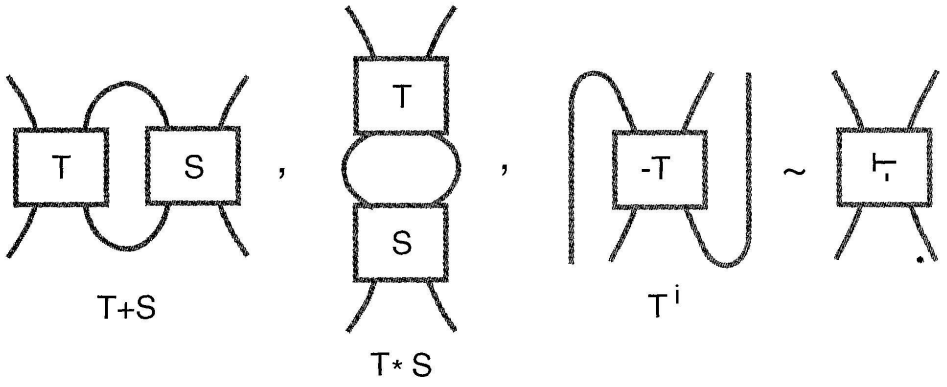


FIGURE 4
Addition, multiplication and inversion of 2-tangles

The *mirror image* of a tangle T, denoted $-T$, is obtained from T by switching all the crossings. So we have $-[n] = [-n]$ and $-\frac{1}{[n]} = \frac{1}{[-n]}$. Finally, *the rotation* of T, denoted T^r, is obtained by rotating T on its plane counterclockwise by 90°, whilst *the inverse* of T, denoted T^i, is defined to be $-T^r$. Thus inversion is accomplished by rotation and mirror image. For

example, $[n]^i = \frac{1}{[n]}$ and $\frac{1}{[n]}^i = [n]$. Note that T^r and T^i are in general not isotopic to T.

Moreover, by joining with simple arcs the two upper and the two lower endpoints of a 2-tangle T, we obtain a knot called the *Numerator* of T, denoted by $N(T)$. Joining with simple arcs each pair of the corresponding top and bottom endpoints of T we obtain the *Denominator* of T, denoted by $D(T)$. We have $N(T) = D(T^r)$ and $D(T) = N(T^r)$. We point out that the numerator closure of the sum of two rational tangles is still a rational knot or link. But the denominator closure of the sum of two rational tangles is not necessarily a rational knot or link, think for example of the sum $\frac{1}{[3]} + \frac{1}{[3]}$.

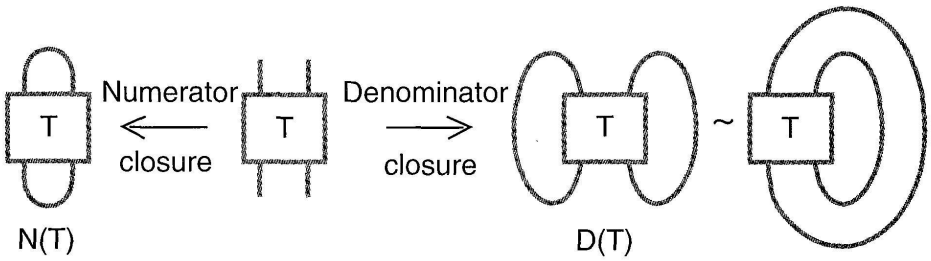


FIGURE 5
The numerator and denominator of a 2-tangle

RATIONAL TANGLE ISOTOPIES. We define now two isotopy moves for rational tangles that play a crucial role in the theory of rational knots and rational tangles.

DEFINITION 1. A *flype* is an isotopy of a 2-tangle T (or a knot or link) applied on a 2-subtangle of the form $[\pm 1]+t$ or $[\pm 1]*t$ as illustrated in Figure 6. A flype fixes the endpoints of the subtangle on which it is applied. A flype shall be called *rational* if the 2-subtangle on which it applies is rational.

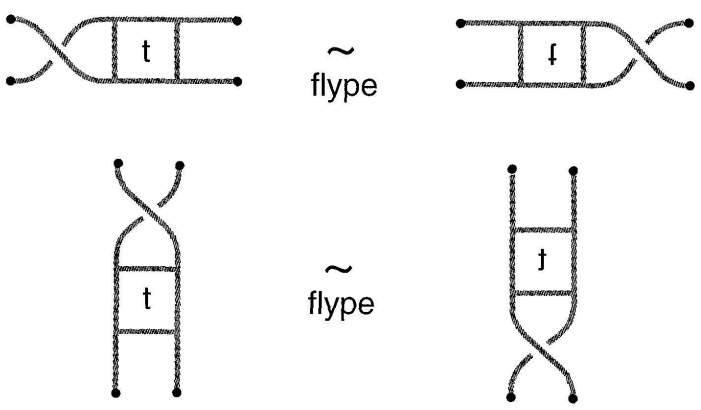


FIGURE 6
The flype moves

We define the *truncation* of a rational tangle to be the result of partially untwisting the tangle. For rational tangles, flypes are of very specific types. Indeed, we have:

Let T be a rational tangle in twist form. Then

(i) *T does not contain any non-rational 2-subtangles.*

(ii) *Every 2-subtangle of T is a truncation of T.*

For a proof of these statements we refer the reader to our paper [15]. As a corollary we have that all flypes of a rational tangle T are rational.

DEFINITION 2. A *flip* is a rotation in space of a 2-tangle by 180°. We say that T^{hflip} is the *horizontal flip* of the 2-tangle T if T^{hflip} is obtained from T by a 180° rotation around a horizontal axis on the plane of T, and T^{vflip} is the *vertical flip* of the tangle T if T^{vflip} is obtained from T by a 180° rotation around a vertical axis on the plane of T. See Figure 7 for illustrations.

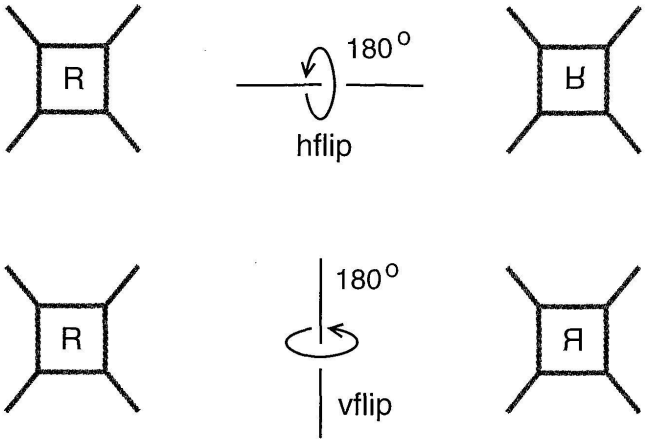


FIGURE 7
The horizontal and the vertical flip

Note that a flip switches the endpoints of the tangle and, in general, a flipped tangle is not isotopic to the original one; the following is a remarkable property of rational tangles:

THE FLIPPING LEMMA. *If T is rational, then:*

(i) $T \sim T^{hflip}$, (ii) $T \sim T^{vflip}$ *and* (iii) $T \sim (T^i)^i = (T^r)^r$.

To see (i) and (ii) we apply induction and a sequence of flypes, see [15] for details. $(T^i)^i = (T^r)^r$ is the tangle obtained from T by rotating it on its plane by 180°, so statement (iii) follows by applying a vertical flip after a horizontal flip. Note that the above statements are obvious for the tangles $[0], [\infty]$, $[n]$ and $\frac{1}{[n]}$. Statement (iii) says that *for rational tangles the inversion is an operation of order* 2. For this reason we shall denote the inverse of a rational tangle T by $1/T$, and hence the rotation of the tangle T will be denoted by $-1/T$. This explains the notation for the tangles $\frac{1}{[n]}$. For arbitrary 2-tangles the inversion is an order 4 operation. Another consequence of the above property is that addition and multiplication by $[\pm 1]$ are commutative.

STANDARD FORM, CONTINUED FRACTION FORM AND CANONICAL FORM FOR RATIONAL TANGLES. Recall that the twists generating the rational tangles could take place between the right, left, top or bottom endpoints of a previously created rational tangle. Using obvious flypes on appropriate subtangles one can always bring the twists all to the right (or all to the left) and to the bottom (or to the top) of the tangle. We shall then say that the rational tangle is in *standard form*. For example Figure 1 illustrates the tangle $(([3] * \frac{1}{[-2]}) + [2])$ in standard form. In order to read out the standard form of a rational tangle in twist form we transcribe it as an algebraic sum using horizontal and vertical twists. For example, Figure 3 illustrates the tangle $((([3] * \frac{1}{[3]}) + [-1]) * \frac{1}{[-4]}) + [2]$ in non-standard form.

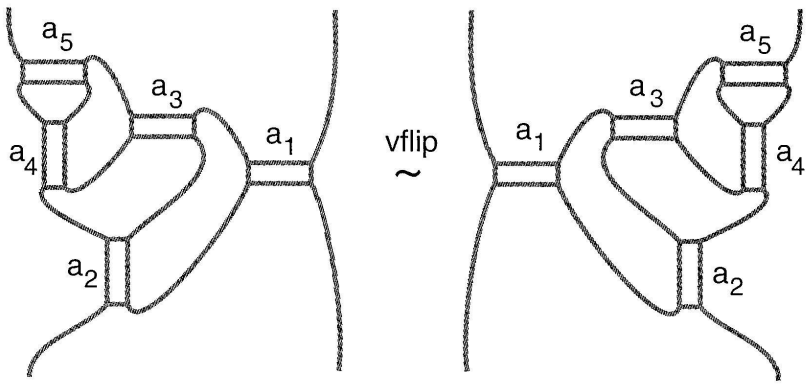


FIGURE 8
The standard representations

Figure 8 illustrates two equivalent (by the Flipping Lemma) ways of representing an abstract rational tangle in standard form: the *standard representation* of a rational tangle. In either illustration the rational tangle begins to twist from the tangle $[a_n]$ ($[a_5]$ in Figure 8), and it untwists from the tangle $[a_1]$. Note that the tangle in Figure 8 has an odd number of sets

of twists ($n = 5$) and this causes $[a_1]$ to be horizontal. If n is even and $[a_n]$ is horizontal then $[a_1]$ has to be vertical.

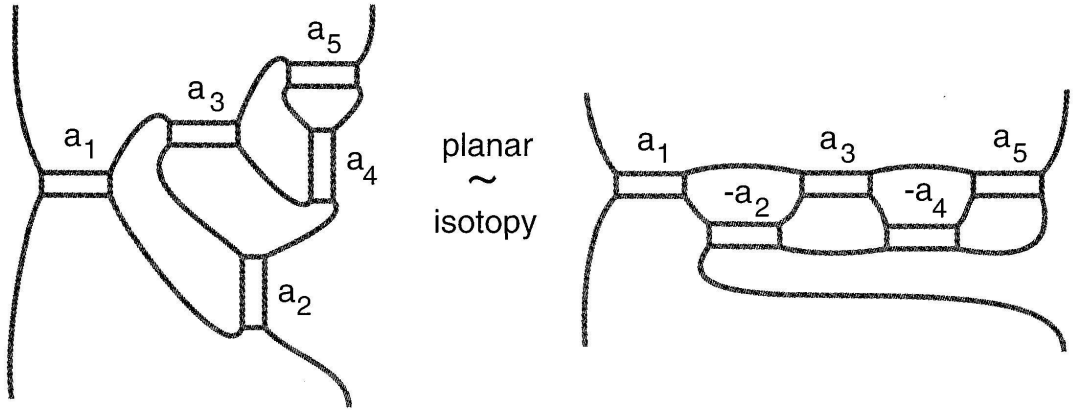


FIGURE 9
The standard and the 3-strand-braid representation

Another way of representing an abstract rational tangle in standard form is illustrated in Figure 9. This is the *3-strand-braid representation*. For an example see Figure 10. As Figure 9 shows, the 3-strand-braid representation is actually a compressed version of the standard representation, so the two representations are equivalent by a planar rotation. The upper row of crossings of the 3-strand-braid representation corresponds to the horizontal crossings of the standard representation and the lower row to the vertical ones. Note that, even though the type of crossings does not change by this planar rotation, we need to draw the mirror images of the even terms, since when we rotate them to the vertical position we obtain crossings of the opposite type in the local tangles. In order to bear in mind this change of the local signs we put on the geometric picture the minuses on the even terms. We shall use both ways of representation for extracting the properties of rational knots and tangles.

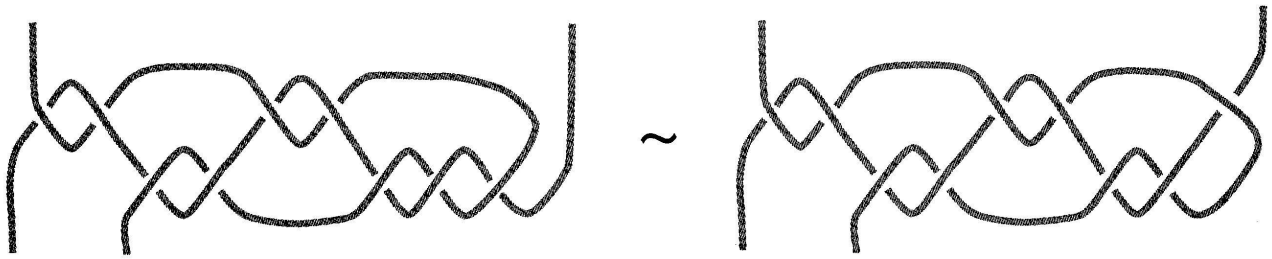


FIGURE 10
The ambiguity of the first crossing

From the above one may associate to a rational tangle diagram in standard form a vector of integers $(a_1, a_2, \ldots, a_n)$, where the first entry denotes the place where the tangle starts unravelling and the last entry where it begins to twist. For example the tangle of Figure 1 is associated to the vector $(2, -2, 3)$, while the tangle of Figure 3 corresponds after a sequence of flypes to the vector $(2, -4, -1, 3, 3)$. The vector associated to a rational tangle diagram is *unique* up to breaking the entry a_n by a unit, i.e. $(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n - 1, 1)$, if $a_n > 0$, and $(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n + 1, -1)$, if $a_n < 0$. This follows from the ambiguity of the very first crossing, see Figure 10. If a rational tangle changes by an isotopy, the associated vector might also change.

REMARK 1. The same ambiguity implies that the number n in the above notation may be assumed to be *odd*. We shall make this assumption for proving Theorems 2 and 3.

The next thing to observe is that a rational tangle in standard form can be described algebraically by a continued fraction built from the integer tangles $[a_1], [a_2], \ldots, [a_n]$ with all numerators equal to 1, namely by an expression of the type:

$$[[a_1], [a_2], \ldots, [a_n]] := [a_1] + \cfrac{1}{[a_2] + \cdots + \cfrac{1}{[a_{n-1}] + \cfrac{1}{[a_n]}}}$$

for $a_2, \ldots, a_n \in \mathbf{Z} - \{0\}$ and n even or odd. We allow $[a_1]$ to be the tangle $[0]$. This expression follows inductively from the equation

$$T * \frac{1}{[n]} = \frac{1}{[n] + \cfrac{1}{T}}.$$

Then a rational tangle is said to be in *continued fraction form*. For example, Figure 1 illustrates the rational tangle $[[2], [-2], [3]]$, while the tangles of Figure 8 and 9 all depict the abstract rational tangle $[[a_1], [a_2], [a_3], [a_4], [a_5]]$.

The tangle equation $T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}}$ implies also that the two simple algebraic operations: *addition of* $[+1]$ *or* $[-1]$ and *inversion* between rational tangles generate the whole class of rational tangles. For $T = [[a_1], [a_2], \ldots, [a_n]]$ the following statements are now straightforward.

1. $T + [\pm 1] = [[a_1 \pm 1], [a_2], \ldots, [a_n]]$,

2. $\dfrac{1}{T} = [[0], [a_1], [a_2], \ldots, [a_n]]$,

3. $-T = [[-a_1], [-a_2], \ldots, [-a_n]]$,

4. $T = [[a_1], [a_2], \ldots, [a_n - 1], [1]]$, if $a_n > 0$,

and $T = [[a_1], [a_2], \ldots, [a_n + 1], [-1]]$, if $a_n < 0$.

A tangle is said to be *alternating* if the crossings alternate from under to over as we go along any component or arc of the weave. Similarly, a knot is *alternating* if it possesses an alternating diagram. We shall see that rational tangles and rational knots are alternating. Notice that, according to the checkerboard shading (see Figure 2 and the corresponding discussion), the only way the weave alternates is if any two adjacent crossings are of the same type, and this propagates to the whole diagram. Thus, *a tangle or a knot diagram with all crossings of the same type is alternating*, and this characterizes alternating tangle and knot diagrams. It is important to note that flypes preserve the alternating structure. Moreover, flypes are the only isotopy moves needed in the statement of the Tait Conjecture for alternating knots. An important property of rational tangles is now the following:

A rational tangle diagram in standard form can be always isotoped to an alternating one.

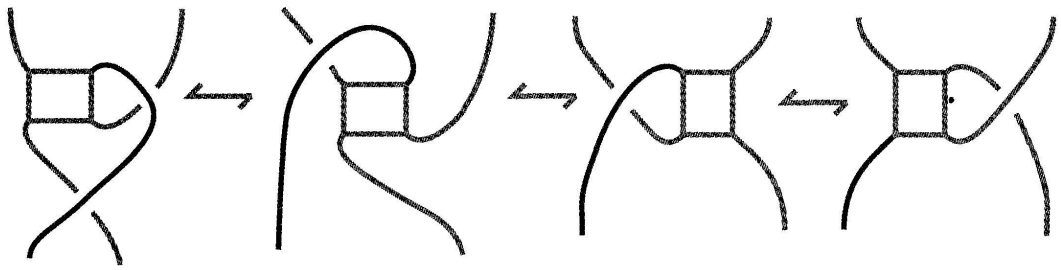


FIGURE 11
Reducing to the alternating form

The process is inductive on the number of crossings and the basic isotopy move is illustrated in Figure 11, see [15] for details. We point out that this isotopy applies to rational tangles in standard form where all the crossings

are on the right and on the bottom. We shall say that a rational tangle $T = [[a_1], [a_2], \ldots, [a_n]]$ is in *canonical form* if T is alternating and n is odd. From Remark 1 we can always assume n to be odd, so in order to bring a rational tangle to the canonical form we just have to apply the isotopy moves described in Figure 11. Note that T alternating implies that the a_i's are all of the same sign.

The alternating nature of the rational tangles will be very useful to us in classifying rational knots and links. It turns out from the classification of alternating knots that *two alternating tangles are isotopic if and only if they differ by a sequence of flypes.* (See [41], [20]. See also [34].) It is easy to see that the closure of an alternating rational tangle is an alternating knot. Thus we have:

Rational knots are alternating, since they possess a diagram that is the closure of an alternating rational tangle.

CONTINUED FRACTIONS AND THE CLASSIFICATION OF RATIONAL TANGLES. From the above discussion it makes sense to assign to a rational tangle in standard form, $T = [[a_1], [a_2], \ldots, [a_n]]$, for $a_1 \in \mathbf{Z}$, $a_2, \ldots, a_n \in \mathbf{Z} - \{0\}$ and n even or odd, the continued fraction

$$F(T) = [a_1, a_2, \ldots, a_n] := a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}},$$

if $T \neq [\infty]$, and $F([\infty]) := \infty = \frac{1}{0}$, as a formal expression. This rational number or infinity shall be called *the fraction of* T. The fraction is a topological invariant of the tangle T. We explain briefly below how to see this.

The subject of continued fractions is of perennial interest to mathematicians. See for example [17], [23], [18], [47]. In this paper we shall only consider continued fractions of the above type, i.e. with all numerators equal to 1. As in the case of rational tangles we allow the term a_1 to be zero. Clearly, the two simple algebraic operations *addition of* $+1$ *or* -1 and *inversion* generate inductively the whole class of continued fractions starting from zero. For any rational number $\frac{p}{q}$ the following statements are really straightforward.

1. There are $a_1 \in \mathbf{Z}$, $a_2, \ldots, a_n \in \mathbf{Z} - \{0\}$ such that $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$,

2. $$\frac{p}{q} \pm 1 = [a_1 \pm 1, a_2, \ldots, a_n],$$

3. $$\frac{q}{p} = [0, a_1, a_2, \ldots, a_n],$$

4. $$-\frac{p}{q} = [-a_1, -a_2, \ldots, -a_n],$$

5. $$\frac{p}{q} = [a_1, a_2, \ldots, a_n - 1, 1], \quad \text{if } a_n > 0,$$

and $$\frac{p}{q} = [a_1, a_2, \ldots, a_n + 1, -1], \quad \text{if } a_n < 0.$$

Property 1 is a consequence of Euclid's algorithm, see for example [17]. Combining the above we obtain the following properties for the tangle fraction.

1. $$F(T + [\pm 1]) = F(T) \pm 1,$$

2. $$F(\frac{1}{T}) = \frac{1}{F(T)},$$

3. $$F(-T) = -F(T).$$

The last ingredient for the classification of rational tangles is the following fact about continued fractions: *Every continued fraction* $[a_1, a_2, \ldots, a_n]$ *can be transformed to a unique canonical form* $[\beta_1, \beta_2, \ldots, \beta_m]$, *where all* β_i*'s are positive or all negative integers and* m *is odd.*

One way to see this is to evaluate the continued fraction and then apply Euclid's algorithm, keeping all remainders of the same sign. There is also an algorithm that can be applied directly to the initial continued fraction to obtain its canonical form. This algorithm works in parallel with the algorithm for the canonical form of rational tangles, see [15] for details.

From the Tait conjecture for alternating rational tangles, from the uniqueness of the canonical form of continued fractions and from the above properties of the fraction we derive that the fraction not only is an isotopy invariant of rational tangles but it also classifies rational tangles. This is the Conway Theorem. See [15] for details of the proof. For the isotopy type of a rational tangle with fraction $\frac{p}{q}$ we shall use the notation $[\frac{p}{q}]$. Finally, it is easy to see the following useful result about rational tangles:

Suppose that $T + [n]$ *is a rational tangle, then* T *is a rational tangle.*

3. THE CLASSIFICATION OF UNORIENTED RATIONAL KNOTS

In this section we shall prove Schubert's theorem for unoriented rational knots. It is convenient to say that reduced fractions p/q and p'/q' are arithmetically equivalent, written $p/q \sim p'/q'$, if $p = p'$ and either $qq' \equiv 1 \bmod p$ or $q \equiv q' \bmod p$. We shall call two rational tangles *arithmetically equivalent* if their fractions are arithmetically equivalent. In this language, Schubert's theorem states that two unoriented rational tangles close to form isotopic knots if and only if they are arithmetically equivalent.

We only need to consider numerator closures of rational tangles, since the denominator closure of a tangle T is simply the numerator closure of its rotate $-\frac{1}{T}$. From the discussions in Section 2 a rational tangle may be assumed to be in continued fraction form and by Remark 1, *the length* of a rational tangle may be assumed to be *odd.* A rational knot is said to be *in standard form, in continued fraction form, alternating or in canonical form* if it is the numerator closure of a rational tangle that is in standard form, in continued fraction form, alternating or in canonical form respectively. By the alternating property of rational knots we may assume all rational knot diagrams to be *alternating*. The diagrams and the isotopies of the rational knots are meant to take place in the 2-sphere and not in the plane.

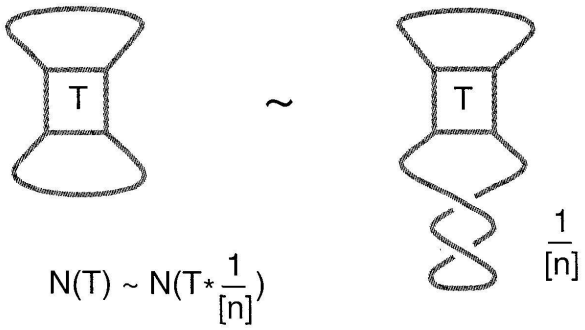


FIGURE 12
Twisting the bottom of a tangle

BOTTOM TWISTS. The simplest instance of two rational tangles being non-isotopic but having isotopic numerators is adding a number of twists at the bottom of a tangle, see Figure 12. Indeed, let T be a rational tangle and let $T * 1/[n]$ be the tangle obtained from T by adding n bottom twists, for any $n \in \mathbf{Z}$. We have $N(T * 1/[n]) \sim N(T)$, but $F(T * 1/[n]) = F(1/([n] + 1/T)) = 1/(n + 1/F(T))$; so, if

$$F(T) = p/q,$$

then

$$F(T * 1/[n]) = p/(np + q),$$

thus the two tangles are not isotopic. If we set $np + q = q'$ we have $q \equiv q' \mod p$, just as Theorem 2 predicts.

Reducing all possible bottom twists of a rational tangle yields a rational tangle with fraction $\frac{P}{Q}$ such that

$$|P| > |Q|.$$

To see this, suppose that we are dealing with $\frac{P}{Q'}$ with $P < Q'$ and both P and Q' positive (we leave it to the reader to fill in the details for Q' negative). Then

$$\frac{P}{Q'} = \frac{1}{\frac{Q'}{P}} = \frac{1}{n + \frac{Q}{P}} = \frac{1}{n + \frac{1}{P/Q}},$$

where

$$Q' = nP + Q \equiv Q \mod P,$$

for n and Q positive and $Q < P$. So, by the Conway Theorem, the rational tangle $[\frac{P}{Q'}]$ differs from the tangle $[\frac{P}{Q}]$ by n bottom twists, and so $N([\frac{P}{Q'}]) \sim N([\frac{P}{Q}])$. Figure 13 illustrates an example of this arithmetic. Note that a tangle with fraction $\frac{P}{Q}$ such that $|P| > |Q|$ always ends with a number of horizontal twists. So, if $T = [[a_1], [a_2], \ldots, [a_n]]$ then $a_1 \neq 0$. If T is in twist form then T will not have any top or bottom twists. We shall say that a rational tangle whose fraction satisfies the above inequality is in *reduced form.*

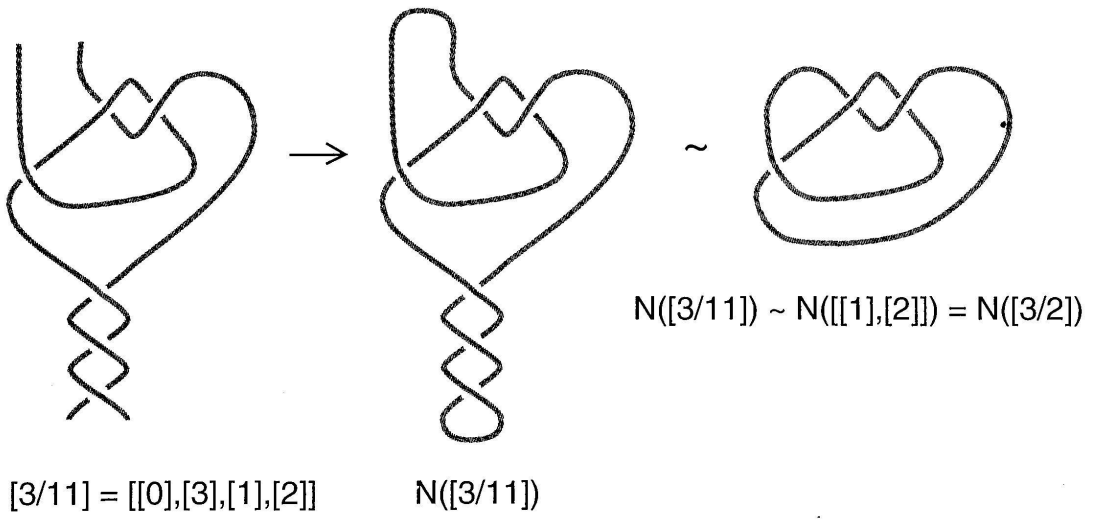


FIGURE 13
Reducing the bottom twists

The proof of Theorem 2 now proceeds in two stages. First, (in 3.1) we look for all possible places where we could cut a rational knot K open to a rational tangle, and we show that all cuts that open K to other rational tangles give tangles arithmetically equivalent to the tangle T. Second, (in 3.2) given two isotopic reduced alternating rational knot diagrams, we have to check that the rational tangles that they open to are arithmetically equivalent. By the solution to the Tait Conjecture these isotopic knot diagrams will differ by a sequence of flypes. So we analyze what happens when a flype is performed on K.

3.1 THE CUTS

Let K be a rational knot that is the numerator closure of a rational tangle T. We will look for all 'rational' cuts on K. In our study of cuts we shall assume that T is in reduced canonical form. The more general case where T is in reduced alternating twist form is completely analogous and we make a remark at the end of the subsection. Moreover, the cut analysis in the case where $a_1 = 0$ is also completely analogous for all cuts with appropriate adjustments. There are three types of rational cuts.

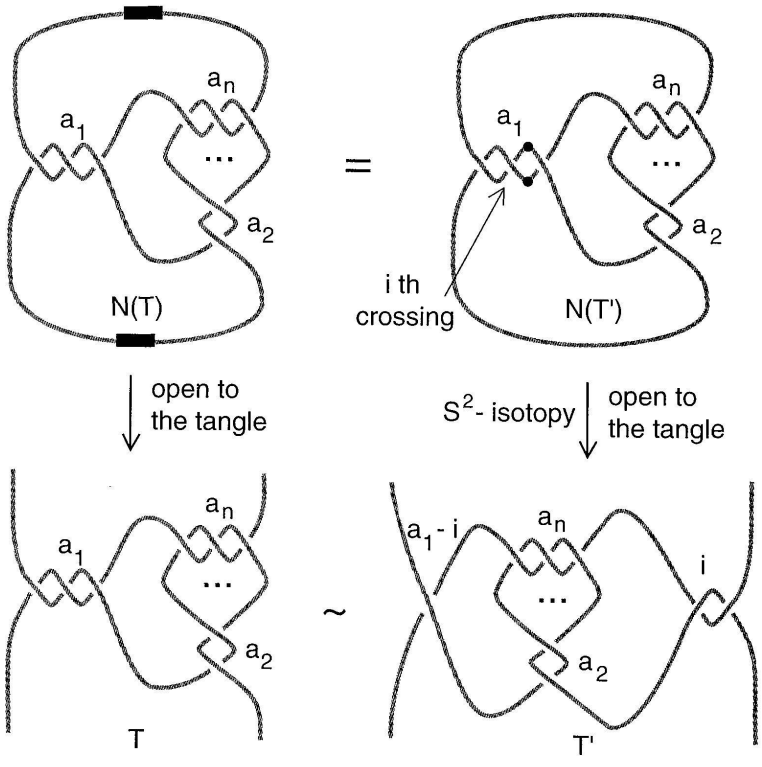


FIGURE 14
Standard cuts

THE STANDARD CUTS. The tangle $T = [[a_1], [a_2], \ldots, [a_n]]$ is said to arise as *the standard cut* on $K = N(T)$. If we cut K at another pair of 'vertical'

points that are adjacent to the ith crossing of the elementary tangle $[a_1]$ (counting from the outside towards the inside of T) we obtain the alternating rational tangle in twist form $T' = [[a_1 - i], [a_2], \ldots, [a_n]] + [i]$. Clearly, this tangle is isotopic to T by a sequence of flypes that send all the horizontal twists to the right of the tangle. See the right hand illustration of Figure 14 for $i = 2$. Thus, by the Conway Theorem, T' will have the same fraction as T. Any such cut on K shall be called a *standard cut* on K.

THE SPECIAL CUTS. A key example of the arithmetic relationship of the classification of rational knots is illustrated in Figure 15. The two tangles $T = [-3]$ and $S = [1] + \frac{1}{[2]}$ are non-isotopic by the Conway Theorem, since $F(T) = -3 = 3/-1$, while $F(S) = 1 + 1/2 = 3/2$. But they have isotopic numerators: $N(T) \sim N(S)$, the left-handed trefoil. Now $-1 \equiv 2 \mod 3$, confirming Theorem 2.

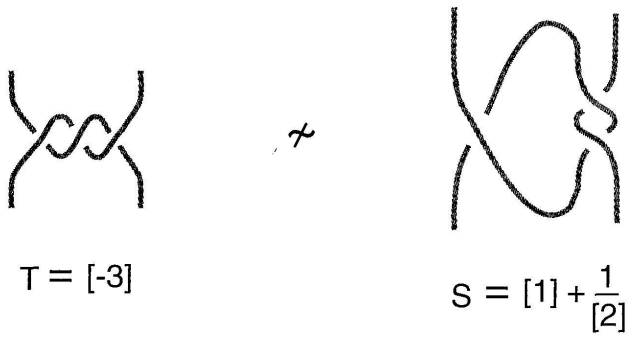


FIGURE 15
An example of the special cut

We now analyze the above example in general. Let $K = N(T)$, where $T = [[a_1], [a_2], \ldots, [a_n]]$. Since T is assumed to be in reduced form, it follows that $a_1 \neq 0$, so T can be written in the form $T = [+1] + R$ or $T = [-1] + R$, and the tangle R is also rational.

The indicated horizontal crossing $[+1]$ of the tangle $T = [+1] + R$, which is the first crossing of $[a_1]$ and the last created crossing of T, may also be seen as a vertical one. So, instead of cutting the diagram K open at the two standard cutpoints to obtain the tangle T, we cut at the two other marked 'horizontal' points on the first crossing of the subtangle $[a_1]$ to obtain a new 2-tangle T' (see Figure 16). T' is clearly rational, since R is rational. The tangle T' is said to arise as *the special cut* on K.

We would like to identify this rational tangle T'. For this reason we first swing the upper arc of K down to the bottom of the diagram in order to free the region of the cutpoints. By our convention for the signs of crossings in

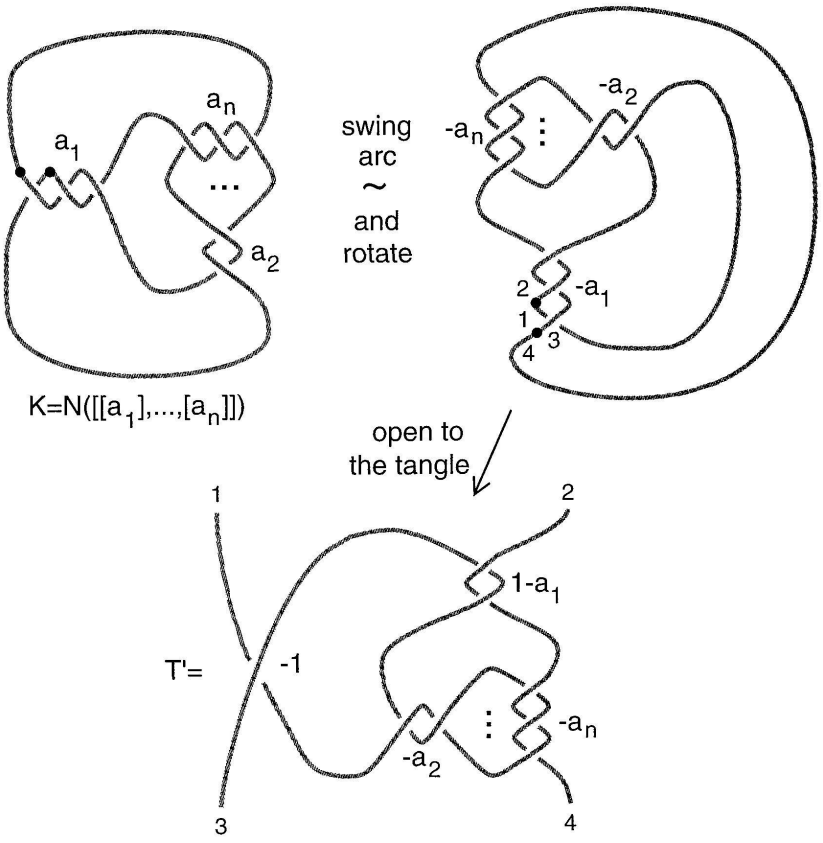


FIGURE 16
Preparing for the special cut

terms of the checkerboard shading, this forces all crossings of T to change sign from positive to negative and vice versa. We then rotate K by 90° on its plane (see right-hand illustration of Figure 16). This forces all crossings of T to change from horizontal to vertical and vice versa. In particular, the marked crossing $[+1]$, that was seen as a vertical one in T, will now look as a horizontal $[-1]$ in T'. In fact, this will be the only last horizontal crossing of T', since all other crossings of $[a_1]$ are now vertical. So, if $T = [[a_1], [a_2], \ldots, [a_n]]$ then $R = [[a_1 - 1], [a_2], \ldots, [a_n]]$ and

$$T' = [[-1], [1 - a_1], [-a_2], \ldots, [-a_n]].$$

Note that if the crossings of K were all of negative type, thus all the a_i's would be negative, the tangle T' would be $T' = [[+1], [-1 - a_1], [-a_2], \ldots, [-a_n]]$. In the example of Figure 15 if we took $R = [-2]$, then $T = [-1] + R$ and $T' = S = [[+1], [+2]]$.

The special cut is best illustrated in Figure 17. We consider the rational knot diagram $K = N([+1] + R)$. (We analyze $N([-1] + R)$ in the same way.) As we see here, a sequence of isotopies and cutting K open allow us to read the new tangle:

$$T' = [-1] - \frac{1}{R}.$$

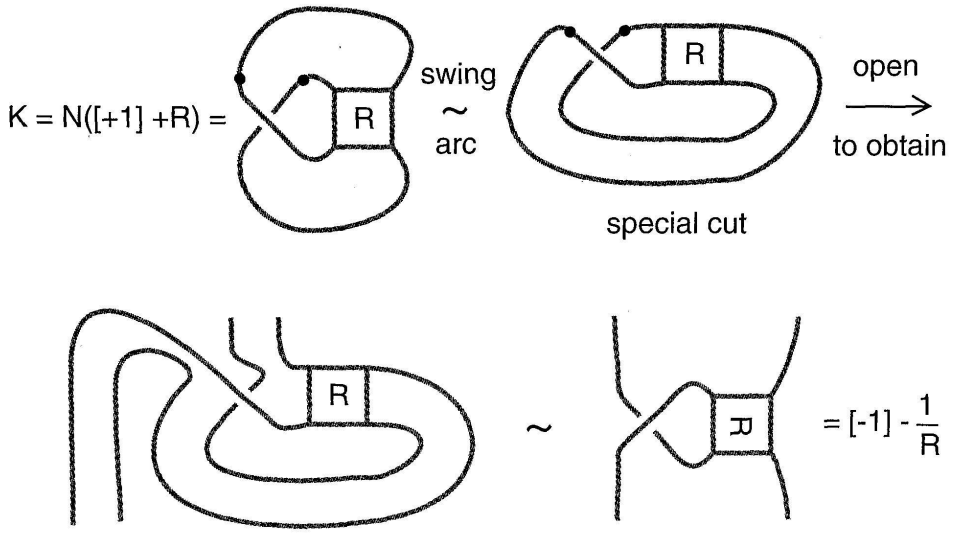


FIGURE 17
The tangle of the special cut

From the above we have $N([+1]+R) \sim N([-1]-\frac{1}{R})$. Let now the fractions of T, R and T' be $F(T) = P/Q$, $F(R) = p/q$ and $F(T') = P'/Q'$ respectively. Then

$$F(T) = F([+1] + R) = 1 + p/q = (p+q)/q = P/Q,$$

while

$$F(T') = F([-1] - 1/R) = -1 - q/p = (p+q)/(-p) = P'/Q'.$$

The two fractions are different, thus the two rational tangles that give rise to the same rational knot are not isotopic. We observe that $P = P'$ and

$$q \equiv -p \mod (p+q) \iff Q \equiv Q' \mod P.$$

This arithmetic equivalence demonstrates another case for Theorem 2. Notice that, although both the bottom twist and the special cut fall into the same arithmetic equivalence, the arithmetic of the special cut is more subtle than the arithmetic of the bottom twist.

If we cut K at the two lower horizontal points of the first crossing of $[a_1]$ we obtain the same rational tangle T'. Also, if we cut at any other pair of upper or lower horizontal adjacent points of the subtangle $[a_1]$ we obtain a rational tangle in twist form isotopic to T'. Such a cut shall be called a *special cut*. See Figure 18 for an example. Finally, we may cut K at any pair of upper or lower horizontal adjacent points of the subtangle $[a_n]$. We shall call this a *special palindrome cut*. We will discuss this case after having analyzed the last type of a cut, the palindrome cut.

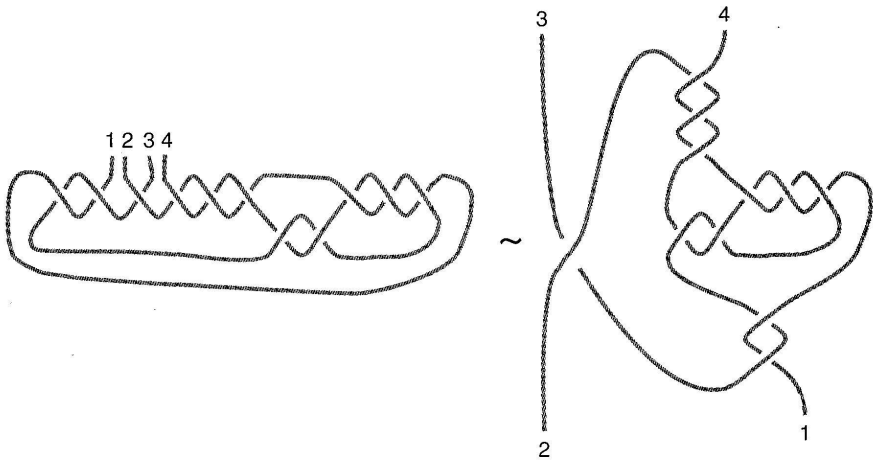


FIGURE 18
A special cut

NOTE. We would like to point out that the horizontal-vertical ambiguity of the last crossing of a rational tangle $T = [[a_1], \ldots, [a_{n-1}], [a_n]]$, which with the special cut on $K = N(T)$ gives rise to the tangle $[[\mp 1], [\pm 1 - a_1], [-a_2], \ldots, [-a_n]]$, is very similar to the horizontal-vertical ambiguity of the first crossing that does not change the tangle and it gives rise to the tangle continued fraction $[[a_1], \ldots, [a_{n-1}], [a_n \mp 1], [\pm 1]]$.

REMARK 2. A special cut is nothing more than the addition of a bottom twist. Indeed, as Figure 19 illustrates, applying a positive bottom twist to the tangle T' of the special cut yields the tangle $S = ([-1] - 1/R) * [+1]$, and we find that if $F(R) = p/q$ then $F([+1] + R) = (p + q)/q$ while $F(([-1] - 1/R) * [+1]) = 1/(1 + 1/(-1 - q/p)) = (p+q)/q$. Thus we see that the fractions of $T = [+1] + R$ and $S = ([-1] - 1/R) * [+1]$ are equal and by the Conway Theorem the tangle S is isotopic to the original tangle T of the standard cut. The isotopy move is nothing but the transfer move of Figure 11. The isotopy is illustrated in Figure 19. Here we used the Flipping Lemma.

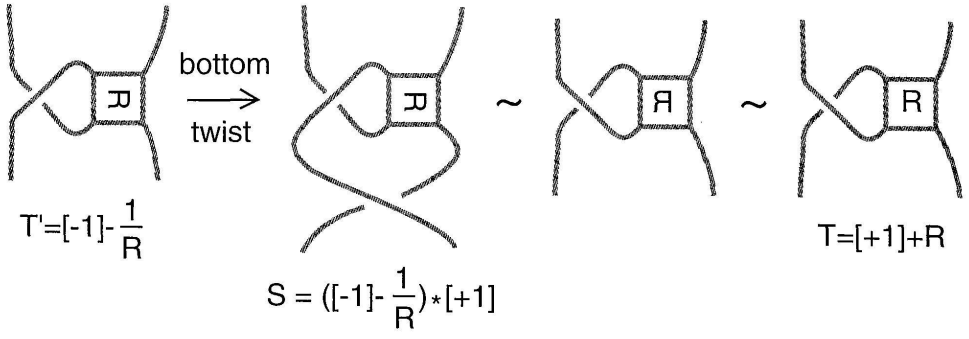


FIGURE 19
Special cuts and bottom twists

THE PALINDROME CUTS. In Figure 20 we see that the tangles

$$T = [[2], [3], [4]] = [2] + \frac{1}{[3] + \frac{1}{[4]}}$$

and

$$S = [[4], [3], [2]] = [4] + \frac{1}{[3] + \frac{1}{[2]}}$$

both have the same numerator closure. This is another key example of the basic relationship given in the classification of rational knots.

In the general case if $T = [[a_1], [a_2], \ldots, [a_n]]$, we shall call the tangle $S = [[a_n], [a_{n-1}], \ldots, [a_1]]$ *the palindrome of* T. Clearly these tangles have the same numerator: $K = N(T) = N(S)$. Cutting open K to yield T is the standard cut, while cutting to yield S shall be called *the palindrome cut* on K.

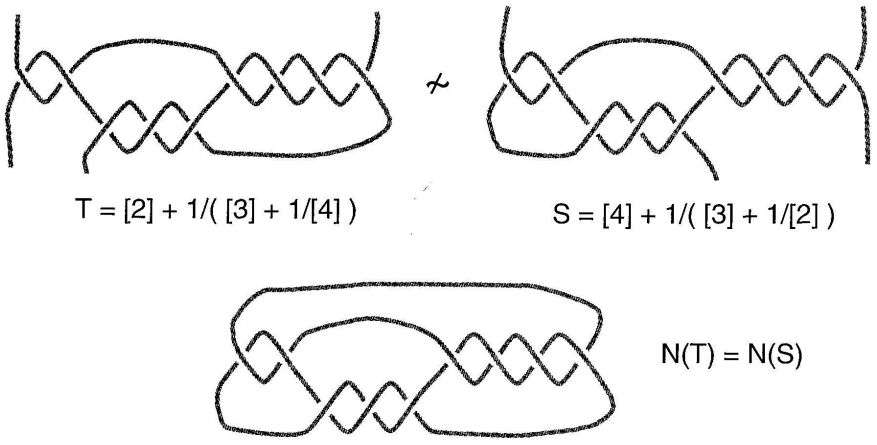


FIGURE 20
An instance of the palindrome equivalence

The tangles in Figure 20 have corresponding fractions

$$F(T) = 2 + \frac{1}{3 + \frac{1}{4}} = \frac{30}{13} \quad \text{and} \quad F(S) = 4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7}.$$

Note that $7 \cdot 13 \equiv 1 \mod 30$. This is the other instance of the arithmetic behind the classification of rational knots in Theorem 2. In order to check the arithmetic in the general case of the palindrome cut we need to generalize this pattern to arbitrary continued fractions and their palindromes (obtained by reversing the order of the terms). Then we have the following

THEOREM 4 (Palindrome Theorem). *Let* $\{a_1, a_2, \ldots, a_n\}$ *be a collection of* n *non-zero integers, and let* $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ *and* $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. *Then* $P = P'$ *and* $QQ' \equiv (-1)^{n+1} \mod P$.

The Palindrome Theorem is a known result about continued fractions. For example see [35] or [16], p. 25, Exercise 2.1.9. We shall give here our proof of this statement. For this we will first present a way of evaluating continued fractions via 2×2 matrices (compare with [11], [18]). This method of evaluation is crucially important in our work in the rest of the paper. Let $\frac{p}{q} = [a_2, a_3, \ldots, a_n]$. Then we have:

$$[a_1, a_2, \ldots, a_n] = a_1 + \frac{1}{\frac{p}{q}} = a_1 + \frac{q}{p} = \frac{a_1 p + q}{p} = \frac{p'}{q'}.$$

Taking the convention that $\left[\binom{p}{q}\right] := \frac{p}{q}$, with our usual conventions for formal fractions such as $\frac{1}{0}$, we can thus write a corresponding matrix equation in the form

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a_1 p + q \\ p \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

We let

$$M(a_i) = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrices $M(a_i)$ are said to be the *generating matrices* for continued fractions, as we have:

LEMMA 1 (Matrix interpretation for continued fractions). *For any sequence of non-zero integers* $\{a_1, a_2, \ldots, a_n\}$ *the value of the corresponding continued fraction is given through the following matrix product*

$$[a_1, a_2, \ldots, a_n] = [M(a_1)M(a_2) \cdots M(a_n) \cdot v]$$

where

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proof. We observe that

$$\left[M(a_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} a_n \\ 1 \end{pmatrix}\right] = a_n = [a_n]$$

and

$$\left[M(a_{n-1})\begin{pmatrix} a_n \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} a_{n-1}a_n + 1 \\ a_n \end{pmatrix}\right] = [a_{n-1}, a_n].$$

Now the lemma follows by induction. □

Proof of the palindrome theorem. We wish to compare $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ and $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. By Lemma 1 we can write

$$\frac{P}{Q} = [M(a_1)M(a_2)\cdots M(a_n)\cdot v] \quad \text{and} \quad \frac{P'}{Q'} = [M(a_n)M(a_{n-1})\cdots M(a_1)\cdot v]\,.$$

Let

$$M = M(a_1)M(a_2)\cdots M(a_n)$$

and

$$M' = M(a_n)M(a_{n-1})\cdots M(a_1)\,.$$

Then $\frac{P}{Q} = [M\cdot v]$ and $\frac{P'}{Q'} = [M'\cdot v]$. We observe that

$$\begin{aligned} M^T &= (M(a_1)M(a_2)\cdots M(a_n))^T = (M(a_n))^T(M(a_{n-1}))^T\cdots(M(a_1))^T \\ &= M(a_n)M(a_{n-1})\cdots M(a_1) = M'\,, \end{aligned}$$

since $M(a_i)$ is symmetric, where M^T is the transpose of T. Thus

$$M' = M^T\,.$$

Let

$$M = \begin{pmatrix} X & Y \\ Z & U \end{pmatrix}\,.$$

In order that the equations $[M\cdot v] = \frac{P}{Q}$ and $[M^T\cdot v] = \frac{P'}{Q'}$ are satisfied it is necessary that $X = P$, $X = P'$, $Z = Q$ and $Y = Q'$. That is, we should have:

$$M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} P & Q \\ Q' & U \end{pmatrix}\,.$$

Furthermore, since the determinant of $M(a_i)$ is equal to -1, we have that

$$\det(M) = (-1)^n\,.$$

Thus

$$PU - QQ' = (-1)^n\,,$$

so that

$$QQ' \equiv (-1)^{n+1} \mod P\,,$$

and the proof of the Theorem is complete. □

REMARK 3. Note in the argument above that the entries of the matrix $M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$ of a given continued fraction $[a_1, a_2, \ldots, a_n] = \frac{P}{Q}$ involve also the evaluation of its palindrome $[a_n, a_{n-1}, \ldots, a_1] = \frac{P}{Q'}$.

Returning now to the analysis of the palindrome cut, we apply Theorem 4 in order to evaluate the fraction of palindrome rational tangles $T = [\frac{P}{Q}]$ and $S = [\frac{P'}{Q'}]$. From the above analysis we have $P = P'$. Also, by our assumption these tangles have continued fraction forms with odd length n, so we have the relation

$$QQ' \equiv 1 \mod P$$

and this is the second of the arithmetic relations of Theorem 2.

If we cut $K = N(T)$ at any other pair of 'vertical' points of the subtangle $[a_n]$ we obtain a rational tangle in twist form isotopic to the palindrome tangle S. Any such cut shall be called a *palindrome cut.*

Having analyzed the arithmetic of the palindrome cuts we can now return to the special palindrome cuts on the subtangle $[a_n]$. These may be considered as special cuts on the palindrome tangle S. So, the fraction of the tangle of such a cut will satisfy the first type of arithmetic relation of Theorem 2 with the fraction of S, namely a relation of the type $q \equiv q' \mod p$, which, consequently, satisfies the second type of arithmetic relation with the fraction of T, namely a relation of the type $qq' \equiv 1 \mod p$. In the end a special palindrome cut will satisfy an arithmetic relation of the second type. This concludes the arithmetic study of the rational cuts.

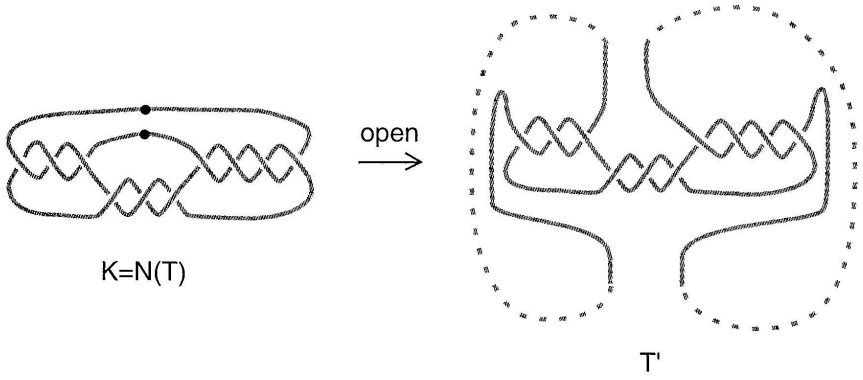


FIGURE 21
A non-rational cut

We now claim that the above listing of the three types of rational cuts is a complete catalog of cuts that can open the link K to a rational tangle: the standard cuts, the special cuts and the palindrome cuts. This is the crux of our proof.

In Figure 21 we illustrate one example of a cut that is not rational. This is a possible cut made in the middle of the representative diagram $N(T)$. Here we see that if T' is the tangle obtained from this cut, so that $N(T') = K$, then $D(T')$ is a connected sum of two non-trivial knots. Hence the denominator $K' = D(T')$ is not prime. Since we know that both the numerator and the denominator of a rational tangle are prime (see [5], p. 91 or [19], Chapter 4, pp. 32–40), it follows that T' is not a rational tangle. Clearly the above argument is generic. It is not hard to see by enumeration that all possible cuts with the exception of the ones we have described will not give rise to rational tangles. We omit the enumeration of these cases.

This completes the proof that all of the rational tangles that close to a given standard rational knot diagram are arithmetically equivalent.

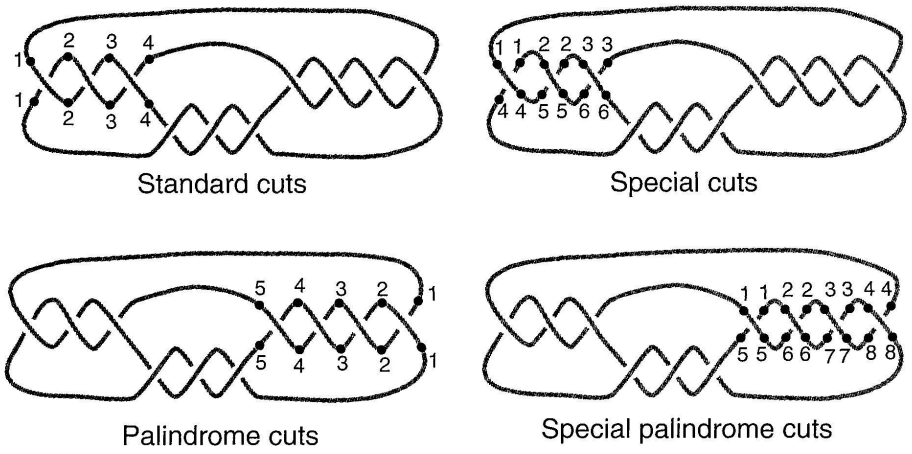


FIGURE 22
Standard, special, palindrome and special palindrome cuts

In Figure 22 we illustrate on a representative rational knot in 3-strand-braid form all the cuts that exhibit that knot as a closure of a rational tangle. Each pair of points is marked with the same number.

REMARK 4. It follows from the above analysis that if T is a rational tangle in twist form, which is isotopic to the standard form $[[a_1], [a_2], \ldots, [a_n]]$, then all arithmetically equivalent rational tangles can arise by any cut of the above types either on the crossings that add up to the subtangle $[a_1]$ or on the crossings of the subtangle $[a_n]$.

3.2 THE FLYPES

Diagrams for knots and links are represented on the surface of the two-sphere, S^2, and then notationally on a plane for purposes of illustration.

Let $K = N(T)$ be a rational link diagram with T a rational tangle in twist form. By an appropriate sequence of flypes (recall Definition 1) we may assume, without loss of generality, that T is alternating and in continued fraction form, i.e. T is of the form $T = [[a_1], [a_2], \ldots, [a_n]]$ with the a_i's all positive or all negative. From the ambiguity of the first crossing of a rational tangle we may assume that n is odd. Moreover, from the analysis of the bottom twists in the previous subsection we may assume that T is in reduced form. Then the numerator $K = N(T)$ will be a reduced alternating knot diagram. This follows from the primality of K.

Let K and K' be two isotopic, reduced, alternating rational knot diagrams. By the Tait Conjecture they will differ by a finite sequence of flypes. In considering how rational knots can be cut open to produce rational tangles, we will examine how the cuts are affected by flyping. We analyze all possible flypes to prove that it is sufficient to consider the cuts on a single alternating reduced diagram for a given rational knot K. Hence the proof will be complete at that point. We need first two definitions and an observation about flypes.

DEFINITION 3. We shall call *region of a flype* the part of the knot diagram that contains precisely the subtangle and the crossing that participate in the flype. The region of a flype can be enclosed by a simple closed curve on the plane that intersects the tangle in four points.

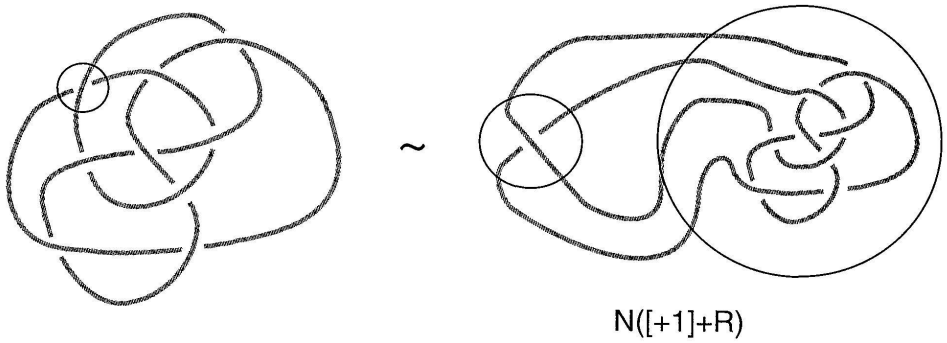


FIGURE 23
Decomposing into $N([\pm 1] + R)$

DEFINITION 4. A *pancake flip* of a knot diagram in the plane is an isotopy move that rotates the diagram by 180° in space around a horizontal or vertical axis on its plane and then it replaces it on the plane. Note that any knot diagram in S^2 can be regarded as a knot diagram in a plane.

In fact, the pancake flip is actually obtained by flypes so long as we allow as background moves isotopies of the diagram in S^2. To see this, note as in Figure 23 that we can assume without loss of generality that the diagram in question is of the form $N([\pm 1] + R)$ for some tangle R not necessarily rational. (Isolate one crossing at the 'outer edge' of the diagram in the plane and decompose the diagram into this crossing and a complementary tangle, as shown in Figure 23.) In order to place the diagram in this form we only need to use isotopies of the diagram in the plane.

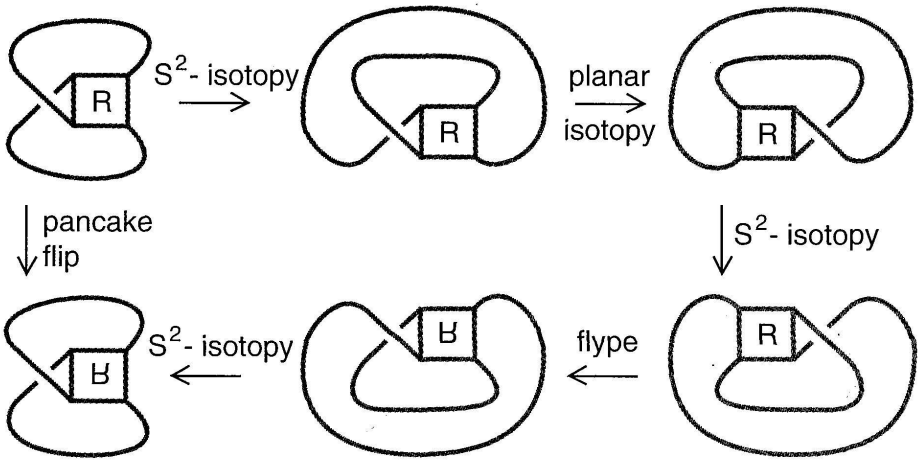


FIGURE 24
Pancake flip

Note now, as in Figure 24, that the pancake flip applied to $N([\pm 1] + R)$ yields a diagram that can be obtained by a combination of a planar isotopy, S^2-isotopies and a flype. (By an *S^2-isotopy* we mean the sliding of an arc around the back of the sphere.) This is valid for R any 2-tangle. We will use this remark in our study of rational knots and links.

We continue with a general remark about the form of a flype in any knot or link in S^2. View Figure 25. First look at parts A and B of this figure. Diagram A is shown as a composition of a crossing and two tangles P and Q. Part B is obtained from a flype of part A, where the flype is applied to the crossing in conjunction with the tangle P. This is the general pattern of the application of a flype. The flype is applied to a composition of a crossing with a tangle, while the rest of the diagram can be regarded as contained within a second tangle that is left fixed under the flyping.

Now look at diagrams C and D. Diagram D is obtained by a flype using Q and a crossing on diagram C. But diagram C is isotopic by a planar isotopy

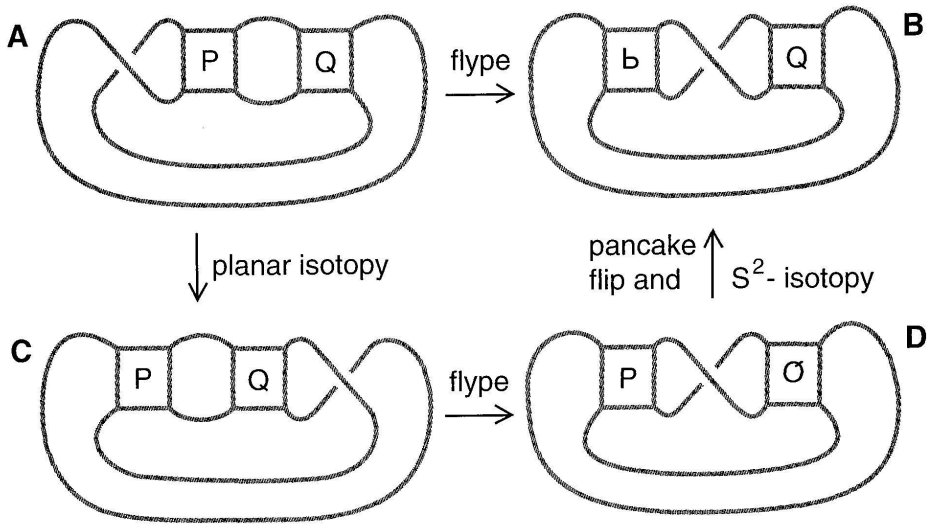


FIGURE 25
The complementary flype

to diagram A, and diagrams B and D are related by a pancake flip (combined with an isotopy that swings two arcs around S^2). Thus we see that:

Up to a pancake flip one can choose to keep either of the tangles P or Q fixed in performing a flype.

Let now $K = N(T)$ and $K' = N(T')$ be two reduced alternating rational knot diagrams that differ by a flype. The rational tangles T and T' are in reduced alternating twist form and without loss of generality T may be assumed to be in continued fraction form. Then, recall from Section 2 that the region of the flype on K can either include a rational truncation of T or some crossings of a subtangle $[a_i]$, see Figure 26. In the first case the two subtangles into which K decomposes are both rational and each will be called the *complementary tangle* of the other. In the second case the flype has really trivial effect and the complementary tangle is not rational, unless $i = 1$ or n.

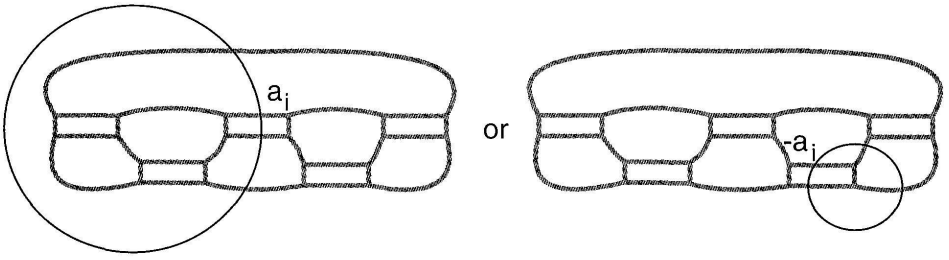


FIGURE 26
Flypes of rational knots

For the cutpoints of T on $K = N(T)$ there are three possibilities:

1. they are outside the region of the flype,
2. they are inside the flyped subtangle,
3. they are inside the region of the flype and outside the flyped subtangle.

If the cutpoints are outside the region of the flype, then the flype is taking place inside the tangle T and so there is nothing to check, since the new tangle is isotopic and thus arithmetically equivalent to T.

We concentrate now on the first case of the region of a flype. If the cutpoints are inside the flyped subtangle then, by Figure 25, this flype can be seen as a flype of the complementary tangle followed by a pancake flip. The region of the flype of the complementary tangle does not contain the cut points, so it is a rational flype that isotopes the tangle to itself. The pancake flip also does not affect the arithmetic, because its effect on the level of the tangle T is simply a horizontal or a vertical flip, and we know that a flipped rational tangle is isotopic to itself.

If the region of the flype encircles a number of crossings of some $[a_i]$ then the cutpoints cannot lie in the region, unless $i = 1$ or n. If the cutpoints do not lie in the region of the flype, there is nothing to check. If they do, then the complementary tangle is isotopic to T, and the pancake flip produces an isotopic tangle.

Finally, if the cutpoints are inside the region of the flype and outside the flyped subtangle, i.e. they are near the crossing of the flype, then there are three cases to check. These are illustrated in Figure 27.

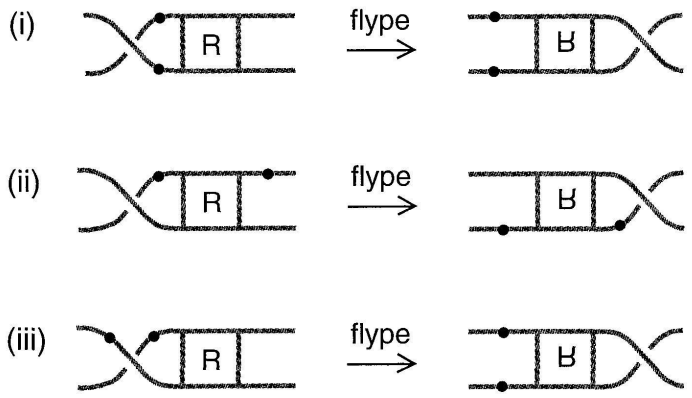


FIGURE 27
Flype and cut interaction

In each of these cases the flype is illustrated with respect to a crossing and a tangle R that is a subtangle of the link $K = N(T)$. Cases (i) and

(ii) are taken care of by the trick of the complementary flype. Namely, as in Figure 25, we transfer the crossing of the flype around S^2. Using this crossing we do a tangle flype of the complementary tangle, then we do a horizontal pancake flip and finally an S^2-isotopy, to end up with the right-hand sides of Figure 27.

In case (iii) we note that after the flype the position of the cut points is outside the region of a flyping move that can be performed on the diagram K' to return to the original diagram K, see Figure 28. This means that after performing the return flype the tangle T' is isotopic to the tangle T''. One can now observe that if the original cut produces a rational tangle, then the cut after the returned flype also produces a rational tangle, and this is arithmetically equivalent to the tangle T. More precisely, the tangle T'' is the result of a special cut on $N(T)$.

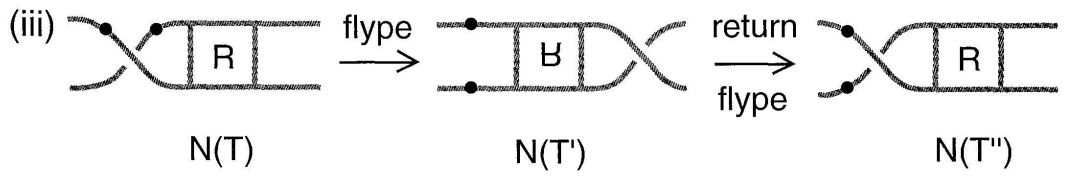


FIGURE 28
Flype and special cut

With the above argument we conclude the proof of the main direction of Theorem 2. From our analysis it follows that:

If $K = N(T)$ is a rational knot diagram with T a rational tangle then, up to bottom twists, any other rational tangle that closes to this knot is available as a cut on the given diagram.

We will now show the converse. We wish to show that if two rational tangles are arithmetically equivalent, then their numerators are isotopic knots. Let T_1, T_2 be rational tangles with $F(T_1) = \frac{p}{q}$ and $F(T_2) = \frac{p}{q'}$, with $|p| > |q|$ and $|p| > |q'|$, and assume first $q\,q' \equiv 1 \mod p$. If $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$, with n odd, and $\frac{p}{q''} = [a_n, a_{n-1}, \ldots, a_1]$ is the corresponding palindrome continued fraction, then it follows from the Palindrome Theorem that $q\,q' \equiv 1 \mod p$. Furthermore, it follows by induction that in a product of the form

$$M(a_1)M(a_2)\cdots M(a_n) = \begin{pmatrix} p & q'' \\ q & u \end{pmatrix}$$

we have that $p > q$ and $p > q''$, $q \geq u$ and $q'' \geq u$ whenever $a_1, a_2, \ldots, a_n$ are positive integers. (With the exception in the case of

$M(1)$ where the first two inequalities are replaced by equalities.) The induction step involves multiplying a matrix in the above form by one more matrix $M(a)$, and observing that the inequalities persist in the product matrix.

Hence, in our discussion we can conclude that $|p| > |q''|$. Since $|p| > |q'|$ and $|p| > |q''|$, it follows that $q' = q''$, since they are both reduced residue solutions of a $\bmod p$ equation with a unique solution. Hence $[a_n, a_{n-1}, \ldots, a_1] = \frac{p}{q'}$, and, by the uniqueness of the canonical form for rational tangles, T_2 has to be:

$$T_2 = [[a_n], [a_{n-1}], \ldots, [a_1]].$$

For these tangles we know that $N(T_1) = N(T_2)$. Let now T_3 be another rational tangle with fraction

$$\frac{p}{q' + kp} = \frac{1}{\frac{q'}{p} + k}.$$

By the Conway Theorem, this is the fraction of the rational tangle

$$\frac{1}{\frac{1}{T_2} + [k]} = T_2 * \frac{1}{[k]}.$$

Hence we have (recall the analysis of the bottom twists):

$$N(\frac{1}{\frac{1}{T_2} + [k]}) \sim N(T_2).$$

Finally, let $F(S_1) = \frac{p}{q}$ and $F(S_2) = \frac{p}{q+kp}$. Then

$$\frac{p}{q + kp} = \frac{1}{\frac{q}{p} + k},$$

which is the fraction of the rational tangle

$$\frac{1}{\frac{1}{S_1} + [k]} = S_1 * \frac{1}{[k]}.$$

Thus

$$N(S_1) \sim N(S_2).$$

The proof of Theorem 2 is now complete. □

We close the section with two remarks.

REMARK 5. In the above discussion about flypes we used the fact that the tangles and flyping tangles involved were rational. One can consider the question of *arbitrary alternating tangles T that close to form links isotopic to a given alternating diagram K*. Our analysis of cuts occurring before and after a flype goes through to show that *for every alternating tangle T, that closes to a diagram isotopic to a given alternating diagram K, there is a cut on the diagram K that produces a tangle that is arithmetically equivalent to T*. Thus it makes sense to consider the collection of tangles that close to an arbitrary alternating link up to this arithmetic equivalence. In the general case of alternating links this shows that on a given diagram of the alternating link we can consider all cuts that produce alternating tangles and thereby obtain all such tangles, up to a certain arithmetical equivalence, that close to links isotopic to K.

Even for rational links there can be more than one equivalence class of such tangles. For example, $N(1/[3] + 1/[3]) = N([-6])$ and $F(1/[3] + 1/[3]) = 2/3$ while $F([-6]) = -6$. Since these fractions have different numerators their tangles (one of which is not rational) lie in different equivalence classes. These remarks lead us to consider the set of arithmetical equivalence classes of altenating tangles that close to a given alternating link and to search for an analogue of Schubert's Theorem in this general setting.

REMARK 6. DNA supercoils, replicates and recombines with the help of certain enzymes. *Site-specific recombination* is one of the ways nature alters the genetic code of an organism, either by moving a block of DNA to another position on the molecule or by integrating a block of alien DNA into a host genome. In [7] it was made possible for the first time to see knotted DNA in an electron micrograph with sufficient resolution to actually identify the topological type of these knots and links. It was possible to design an experiment involving successive DNA recombinations and to examine the topology of the products. In [7] the knotted DNA produced by such successive recombinations was consistent with the hypothesis that all recombinations were of the type of a positive half twist as in $[+1]$. Then D.W. Sumners and C. Ernst [9] proposed a *tangle model for successive DNA recombinations* and showed, in the case of the experiments in question, that there was no other topological possibility for the recombination mechanism than the positive half twist $[+1]$. Their work depends essentially on the classification theorem for

rational knots. This constitutes a unique use of topological mathematics as a theoretical underpinning for a problem in molecular biology.

4. Rational knots and their mirror images

In this section we give an application of Theorem 2. An unoriented knot or link K is said to be *achiral* if it is topologically equivalent to its mirror image $-K$. If a link is not equivalent to its mirror image then it is said be *chiral*. One then can speak of the *chirality* of a given knot or link, meaning whether it is chiral or achiral. Chirality plays an important role in the applications of knot theory to chemistry and molecular biology. In [8] the authors find an explicit formula for the number of achiral rational knots among all rational knots with n crossings. It is interesting to use the classification of rational knots and links to determine their chirality. Indeed, we have the following well-known result (for example see [35] and [16], p. 24, Exercise 2.1.4; compare also with [31]):

Theorem 5. *Let $K = N(T)$ be an unoriented rational knot or link, presented as the numerator of a rational tangle T. Suppose that $F(T) = p/q$ with p and q relatively prime. Then K is achiral if and only if $q^2 \equiv -1 \mod p$. It follows that the tangle T has to be of the form $[[a_1], [a_2], \ldots, [a_k], [a_k], \ldots, [a_2], [a_1]]$ for any integers $a_1, \ldots, a_k$.*

Note that in this description we are using a representation of the tangle with an even number of terms. The leftmost twists $[a_1]$ are horizontal, thus $|p| > |q|$. The rightmost starting twists are then vertical.

Proof. With $-T$ the mirror image of the tangle T, we have that $-K = N(-T)$ and $F(-T) = p/(-q)$. If K is isotopic to $-K$, it follows from the classification theorem for rational knots that either $q(-q) \equiv 1 \mod p$ or $q \equiv -q \mod p$. Without loss of generality we can assume that $0 < q < p$. Hence $2q$ is not divisible by p and therefore it is not the case that $q \equiv -q \mod p$. Hence $q^2 \equiv -1 \mod p$.

Conversely, if $q^2 \equiv -1 \mod p$, then it follows from the Palindrome Theorem that *the continued fraction expansion of p/q has to be palindromic with an even number of terms.* To see this, let $p/q = [c_1, \ldots, c_n]$ with n even, and let $p'/q' = [c_n, \ldots, c_1]$. The Palindrome theorem tells us that $p' = p$ and that $qq' \equiv -1 \mod p$. Thus we have that both q and q' satisfy

the equation $qx \equiv -1 \mod p$ and both q and q' are between 1 and $p-1$. Since this equation has a unique solution in this range, we conclude that $q = q'$. It follows at once that the continued fraction sequence for p/q is symmetric.

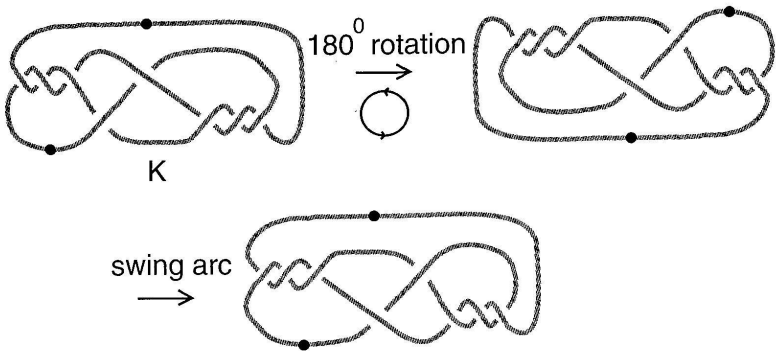


FIGURE 29
An achiral rational link

It is then easy to see that the corresponding rational knot or link $K = N(T)$ is equivalent to its mirror image. One rotates K by 180° in the plane and swings an arc, as Figure 29 illustrates. The point is that the crossings of the second row of the tangle T, that are seemingly crossings of opposite type than the crossings of the upper row, become after the turn crossings of the upper row, and so the types of crossings are switched. This completes the proof. □

5. ON CONNECTIVITY

We shall now introduce the notion of *connectivity* and we shall relate it to the fraction of unoriented rational tangles. We shall say that an unoriented rational tangle has *connectivity type* [0] if the NW end arc is connected to the NE end arc and the SW end arc is connected to the SE end arc. These are the same connections as in the tangle [0]. Similarly, we say that the tangle has *connectivity type* [∞] *or* [1] if the end arc connections are the same as in the tangles [∞] and [+1] (or equivalently [−1]) respectively. The basic connectivity patterns of rational tangles are exemplified by the tangles [0], [∞] and [+1]. We can represent them iconically by

$$[0] = \asymp$$

$$[\infty] = ><$$

$$[1] = \chi$$

For connectivity we are only concerned with the connection patterns of the four end arcs. Thus $[n]$ has connectivity χ whenever n is odd, and connectivity $\asymp$ whenever n is even.

Note that connectivity type [0] yields two-component rational links, whilst type [1] or $[\infty]$ yields one-component rational links. Also, adding a bottom twist to a rational tangle of connectivity type [0] will not change the connectivity type of the tangle, while adding a bottom twist to a rational tangle of connectivity type $[\infty]$ will switch the connectivity type to [1] and vice versa.

We need to keep an accounting of the connectivity of rational tangles in relation to the parity of the numerators and denominators of their fractions. We adopt the following notation: e stands for *even* and o for *odd*. The *parity of a fraction* p/q is defined to be the ratio of the parities (e or o) of its numerator and denominator p and q. Thus the fraction $2/3$ is of parity e/o. The tangle [0] has fraction $0 = 0/1$, thus parity e/o. The tangle $[\infty]$ has formal fraction $\infty = 1/0$, thus parity o/e. The tangle $[+1]$ has fraction $1 = 1/1$, thus parity o/o, and the tangle $[-1]$ has fraction $-1 = -1/1$, thus parity o/o. We then have the following result.

THEOREM 6. *A rational tangle T has connectivity type $\asymp$ if and only if its fraction has parity e/o. T has connectivity type $><$ if and only if its fraction has parity o/e. Finally, T has connectivity type χ if and only if its fraction has parity o/o.*

Proof. Since $F([0]) = 0/1$, $F([\pm 1]) = \pm 1/1$ and $F([\infty]) = 1/0$, the theorem is true for these elementary tangles. It remains to show by induction that it is true for any rational tangle T. Note how connectivity type behaves under the addition and product of tangles:

$$\chi + \chi = \asymp$$
$$\chi + \asymp = \chi$$
$$\asymp + \asymp = \asymp$$
$$\chi + >< = ><$$
$$\asymp + >< = ><$$
$$>< + >< = ><>< = \delta ><$$

$$\chi * \chi = ><$$
$$\chi * \asymp = \asymp$$
$$\asymp * \asymp = \delta \asymp$$
$$\chi * >< = \chi$$
$$\asymp * >< = \asymp$$
$$>< * >< = ><$$

The symbol δ stands for the value of a loop formed. Now any rational tangle can be built from [0] or [∞] by successive addition or multiplication with [± 1]. Thus, from the point of view of connectivity, it suffices to show that $[T]+[\pm 1]$ and $[T]*[\pm 1]$ satisfy the theorem whenever $[T]$ satisfies the theorem. This is checked by comparing the connectivity identities above with the parity of the fractions. For example, in the case

$$\chi + \chi = \asymp \quad \text{we have } o/o + o/o = e/o$$

exactly in accordance with the connectivity identity. The other cases correspond as well, and this proves the theorem by induction. □

COROLLARY 1. *For a rational tangle T the link $N(T)$ has two components if and only if T has fraction $F(T)$ of parity e/o.*

Proof. By the Theorem we have $F(T)$ has parity e/o if and only if T has connectivity type $\asymp$. It follows at once that $N(T)$ has two components. □

Another useful fact is that the components of a rational link are individually unknotted embeddings in three dimensional space. To see this, observe that upon removing one strand of a rational tangle, the other strand is an unknotted arc.

6. THE ORIENTED CASE

Oriented rational knots and links are numerator (and denominator) closures of oriented rational tangles. Rational tangles are oriented by choosing an orientation for each strand of the tangle. Two oriented rational tangles are *isotopic* if they are isotopic as unoriented tangles via an isotopy that carries the orientation of one tangle to the orientation of the other. Since the end arcs of a tangle are fixed during a tangle isotopy, this means that isotopic tangles must

have identical orientations at their end arcs. Thus, *two oriented tangles are isotopic if and only if they are isotopic as unoriented tangles and they have identical orientations at their end arcs.* It follows that a given unoriented rational tangle can always yield non-isotopic oriented rational tangles, for different choices of orientation of one or both strands.

In order to compare oriented rational knots via rational tangles we are only interested in orientations that yield consistently oriented knots upon taking the numerator closure. This means that the two top end arcs have to be oriented one inward and the other outward. Same for the two bottom end arcs.

Reversing the orientation of one strand of an oriented rational tangle that gives rise to a two-component link will usually yield non-isotopic oriented rational links. Figure 30 illustrates an example of non-isotopic oriented rational links, which are isotopic as unoriented links. But reversing a single strand may also yield isotopic oriented rational links. This will be the subject of the next section.

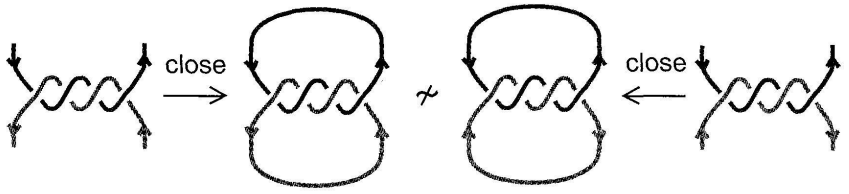


FIGURE 30
Non-isotopic oriented rational links

An oriented knot or link is said to be *invertible* if it is oriented isotopic to its inverse, i.e. the link obtained from it by reversing the orientation of each component. We can obtain the inverse of a rational link by reversing the orientation of both strands of the oriented rational tangle of which it is the numerator. It is easy to see that any rational knot or link is invertible. See the example on the right-hand side of Figure 31.

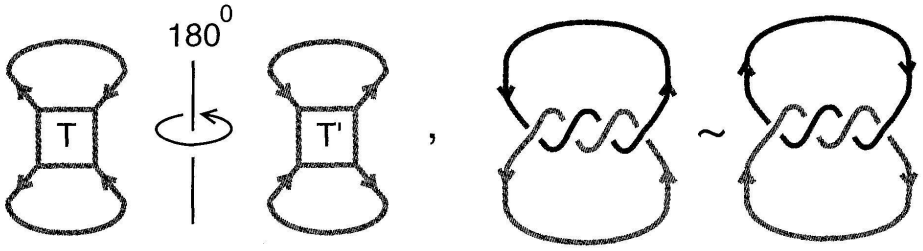


FIGURE 31
Isotopic oriented rational knots and links

LEMMA 2. *Rational knots and links are invertible.*

Proof. Let $K = N(T)$ be an oriented rational knot or link with T an oriented rational tangle. We do a vertical 180°-rotation in space, as the left-hand side of Figure 31 demonstrates. This rotation is a vertical flip for the rational tangle T. Let T' denote the result of the vertical flip of the tangle T. The resulting oriented knot $K' = N(T')$ is oriented isotopic to K, while the orientation of T' is the opposite of that of T on both strands, and thus on all end arcs. But as we have already noted T is isotopic to its vertical flip as unoriented tangles, thus they will have the same fraction. It follows that T' can be isotoped to T through an (unoriented) isotopy that leaves the external strands fixed. Therefore, the result of the vertical 180°-rotation is the tangle T but with all orientations reversed. Thus K' is the inverse of K, and from the above K is oriented isotopic to its inverse. □

Using this observation we conclude that, as far as the study of oriented rational knots is concerned, *all oriented rational tangles may be assumed to have the same orientation for their two upper end arcs.* Indeed, if the orientations of the two upper end arcs are opposite of the fixed ones we do a vertical flip to obtain the orientation pattern that agrees with our convention. We fix this orientation to be downward for the NW end arc and upward for the NE end arc, as in the examples of Figure 30 and as illustrated in Figure 32.

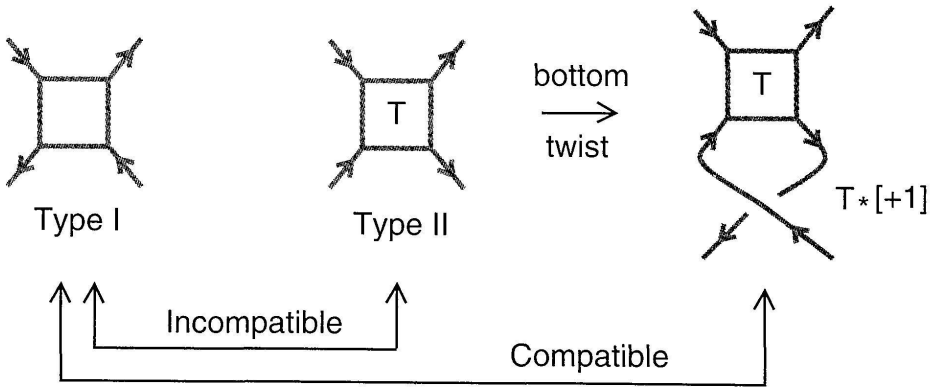


FIGURE 32
Compatible and incompatible orientations

Thus we may reduce our analysis to two basic types of orientation for the four end arcs of a rational tangle. We shall call an oriented rational tangle *of type I* if the SW arc is oriented downward and the SE arc is oriented upward, and *of type II* if the SW arc is oriented upward and the SE arc is oriented downward, see Figure 32. From the above remarks any tangle is of type I or

type II. Two tangles are said to be *compatible* it they are both of type I or both of type II and *incompatible* if they are of different types.

In order to classify oriented rational knots, seen as numerator closures of oriented rational tangles, we will always compare compatible rational tangles.

While the connectivity type of unoriented rational tangles may be [0], [∞] or [1], note that an oriented rational tangle of type I will have connectivity type [0] or [∞] and an oriented rational tangle of type II will have connectivity type [0] or [1].

BOTTOM TWIST BASICS. If two oriented tangles are incompatible, adding a single half twist at the bottom of one of them yields a new pair of compatible tangles, as Figure 32 illustrates. Note also that adding such a twist, although it changes the tangle, does not change the isotopy type of the numerator closure. Thus, up to bottom twists, we are always able to compare oriented rational tangles of the same orientation type. Further, note that if we add a positive bottom twist to an oriented rational tangle T with fraction $F(T) = p/q$ we obtain the incompatible tangle $T' = T * [+1]$ with fraction $F(T') = 1/(1 + 1/F(T)) = p/(p + q)$. Similarly, if we add a negative twist we obtain the incompatible tangle $T'' = T * [-1]$ with fraction $F(T'') = 1/(-1 + 1/F(T)) = p/(-p + q)$. It is worth noting here that the tangles T' and T'' are compatible and $p+q \equiv (-p+q) \mod 2p$, confirming the Oriented Schubert Theorem.

Schubert [31] proved his version of Theorem 3 by using the 2-bridge representation of rational knots and links. We give a tangle-theoretic proof of Schubert's Oriented Theorem, based upon the combinatorics of the unoriented case and then analyzing how orientations and fractions are related.

In our statement of Theorem 3 in the introduction we restricted the denominators of the fractions to be odd. This is a restriction made for the purpose of comparison of tangles. There is no loss of generality, as will be seen when we analyze the palindrome case in the proof at the end of this section. What happens is this: In the case of p odd and only one of q and q' even, one finds that the corresponding tangles are incompatible. We can then compare them by adding a bottom twist to one of the tangles. Adding this twist, the even denominator is replaced by an odd denominator. In the case where p is odd and both q and q' are even, one finds that the corresponding tangles are compatible. In this case, we add a twist at the bottom of each

tangle to preserve the hypothesis that both denominators are odd. This extra twisting yields compatible tangles and a successful comparison.

The strategy of our proof is as follows. Consider an oriented rational knot or link diagram K given in standard form as $N(T)$, where T is a rational tangle in continued fraction form. Our previous analysis tells us that, up to bottom twists, any other rational tangle that closes to this knot is available as a cut on the given diagram. If two rational tangles close to give K as an unoriented rational knot or link, then there are orientations on these tangles, induced from K, so that the oriented tangles close to give K as an oriented knot or link. Two tangles so produced may or may not be compatible. However, adding a bottom twist to one of two incompatible tangles results in two compatible tangles. *It is this possible twist difference that gives rise to the change from modulus p in the unoriented case to the modulus $2p$ in the oriented case.*

We now analyze when, comparing with the original standard cut, another cut produces a compatible or incompatible tangle. See Figure 34 for an example illustrating the compatibility of orientations in the case of the palindrome cut. Note that reducing all possible bottom twists implies $|p| > |q|$ for both tangles, if the two reduced tangles that we compare each time are compatible, or for only one, if they are incompatible. Recall Figure 12 and the related analysis for the basic arithmetic of the bottom twists.

EVEN BOTTOM TWISTS. The simplest instance of the classification of oriented rational knots is adding an *even number of twists* at the bottom of an oriented rational tangle T. We then obtain a compatible tangle $T * 1/[2n]$, and $N(T * 1/[2n]) \sim N(T)$. If now $F(T) = p/q$, then $F(T * 1/[2n]) = F(1/([2n] + 1/T)) = 1/(2n + 1/F(T)) = p/(2np + q)$. Hence, if we set $2np + q = q'$ we have

$$q \equiv q' \mod 2p,$$

just as Theorem 3 predicts.

We then have to compare the special cut and the palindrome cut with the standard cut. Here also, the special cut is the easier to see whilst the palindrome cut requires a more sophisticated analysis. Figure 17 explained how to obtain the unoriented tangle of the special cut. Moreover, by Remark 2, adding a bottom twist to the tangle of the special cut yields a tangle isotopic to the tangle of the standard cut.

Figure 33 demonstrates that the special cut yields oriented incompatible tangles. More precisely, in the case of the special cut we are presented with the general fact that for any tangle R, $N([+1] + R)$ and $N([-1] - 1/R)$

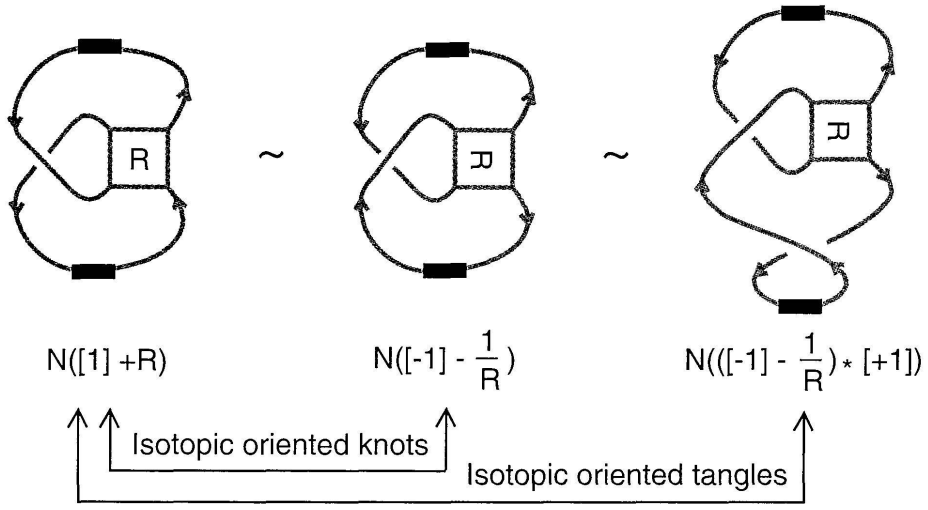


FIGURE 33
The oriented special cut yields incompatible tangles

are unoriented isotopic. With orientations coming from the cut we find that $S = [+1] + R$ and $S' = [-1] - 1/R$ are incompatible. Adding a bottom twist yields oriented compatible tangles, which from the above are isotopic. So, there is nothing to check and the Oriented Schubert Theorem is verified in the strongest possible way for the oriented special cut.

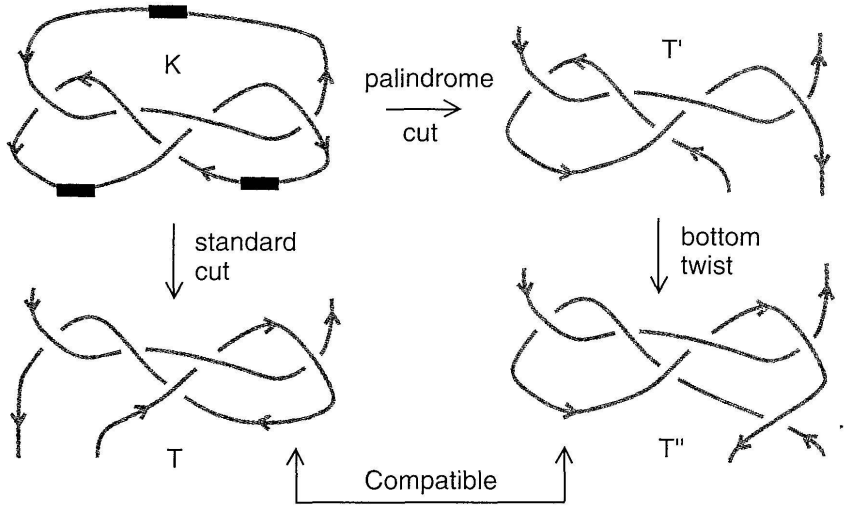


FIGURE 34
Oriented standard cut and palindrome cut

We are left to examine the case of the palindrome cut. In order to analyze this case, we must understand when the standard cut and the palindrome cut are compatible or incompatible. Then we must compare their respective fractions. Figure 34 illustrates how compatibility is obtained by using a bottom twist, in the case of a palindrome cut. In this example we illustrate the standard

and palindrome cuts on the oriented rational knot $K = N(T) = N(T')$ where $T = [[2], [1], [2]]$ and T' its palindrome. As we can see, the two cuts place incompatible orientations on the tangles T and T'. Adding a twist at the bottom of T' produces a tangle $T'' = T' * [-1]$ that is compatible with T. Now we compute $F(T) = F(T') = 8/3$ and $F(T'') = F(T' * [-1]) = 8/-5$ and we notice that $3 \cdot (-5) \equiv 1 \mod 16$, as Theorem 3 predicts.

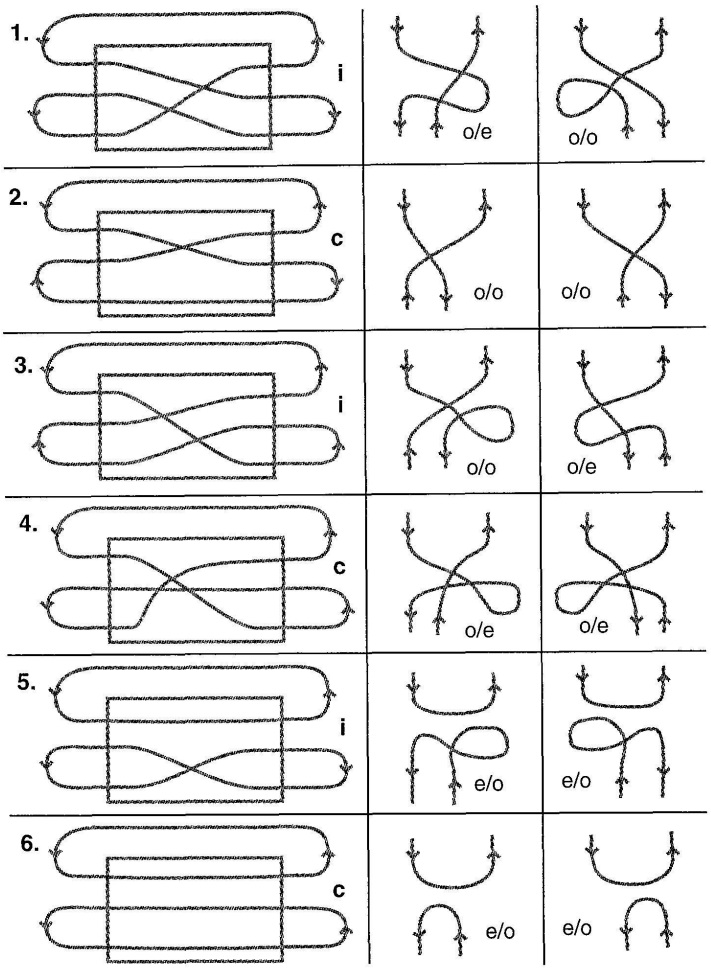


FIGURE 35

The six connection structures, compatibility and parity of the palindrome cut

The study of the compatibility or not of the palindrome cut involves a deeper analysis along the lines of Theorem 6. With the issues of connectivity in place we can begin to analyze the different connectivities and parities in the standard and palindrome cuts on a rational knot or link in standard 3-strand-braid representation. See Figure 35. In this figure we have enumerated the six connection structures for a 3-strand braid (corresponding to the six permutations of three points) with plat closures (of the braid augmented by an extra strand) corresponding to oriented rational knots and links. These closed connection patterns shall be called *connectivity charts*. We then show

corresponding to each connectivity chart the related standard and palindrome cuts and the connectivity and parity of the corresponding tangles. Compatibility or incompatibility of these tangles, specified by an 'i' or 'c', can be read from the oriented diagrams in the figure.

Proof of the palindrome cut. It suffices to verify the Theorem in all cases of the comparison of standard and palindrome cuts on a rational knot K in continued fraction form. We can assume that $K = N([[a_1], \ldots, [a_n]])$ with n odd. Then the tangle $T = [[a_1], \ldots, [a_n]]$ is, by construction, the standard cut on K. We know that the matrix product

$$M = M(a_1)M(a_2)\cdots M(a_n) = \begin{pmatrix} p & q' \\ q & u \end{pmatrix}$$

encodes the fractions of T and its palindrome $T' = [[a_n], \ldots, [a_1]]$, with $F(T) = p/q$ and $F(T') = p/q'$. Note that, since $Det(M) = -1$, we have the formula

$$q\,q' = 1 + up$$

relating the denominators of these fractions.

CASE 1. p ODD, PART A:

If only one of q or q' is even (parts 1 and 3 of Figure 35), then the fact that $q\,q' = 1 + up$ implies the parity equation $e = 1 + uo$, hence u *is odd.* Now refer to Figure 35 and note that the standard and palindrome cuts are incompatible in both cases 1 and 3. (The cases are $\{o/e, o/o\}$ and $\{o/o, o/e\}$.) In order to obtain compatibility, add a bottom twist to the cut with even denominator. Without loss of generality, we may assume that q' is even, so that we will compare p/q and $p/(p + q')$. Note that

$$q(p + q') = q\,p + q\,q' = q\,p + 1 + up = 1 + (u + q)p\,.$$

Since u is odd and q is odd, it follows that $(u+q)$ is even. Hence, $q(p+q') \equiv 1 \mod 2p$, proving Theorem 3 in this case.

CASE 1. p ODD, PART B:

Now suppose that *both q and q' are even.* We are in part 4 of Figure 35 and the two cuts are compatible. Therefore we apply a bottom twist to each cut giving the fractions $p/(p + q)$ and $p/(p + q')$ for comparison. Note that

$$(p + q)(p + q') = p^2 + q\,p + q'p + q\,q' = 1 + (p + q + q' + u)\,p$$

and we have the parity equation

$$p + q + q' + u = o + e + e + o = e .$$

Hence $(p + q)(p + q') \equiv 1 \mod 2p$ verifying the Theorem in this case.

CASE 1. p ODD, PART C:

Finally (for Case 1) suppose that q *and* q' *are both odd.* Then the parity equation corresponding to $qq' = 1 + up$ is

$$o = 1 + uo .$$

Hence u is even so that $qq' \equiv 1 \mod 2p$. We are in part 2 of Figure 35, and the standard and palindrome cuts are compatible. This is in accord with the congruence above, hence the Theorem is verified in this case.

CASE 2. p EVEN:

Now we assume that p is even. This corresponds to parts 5 and 6 in Figure 35 (two components). In part 5 the cuts are compatible, while in part 6 the cuts are incompatible. In either case, both q and q' are odd so that the fractions p/q and p/q' both have the parity e/o. The equation $qq' = 1 + up$ has corresponding parity equation $o = 1 + ue$, and u can be either even or odd. In order to accomplish the proof of Case 2 we will show that

1. u is even if and only if the standard and palindrome cuts are compatible.
2. u is odd if and only if the standard and palindrome cuts are incompatible.

We prove these statements by induction on the number of terms in the continued fraction $[a_1, \ldots, a_n]$. The induction step consists in adding two more terms to the continued fraction (thereby maintaining an odd number of terms). That is, we shall examine a continued fraction in the form $T_{n+2} = [a_1, \ldots, a_{n+2}]$ that is *given* to be in cases 5 or 6 of Figure 35. See Figure 36. In Figure 36 the numbers that label the diagrams refer to the cases in Figure 35. We consider the structure of the "predecessor" $T_n = [a_1, \ldots, a_n]$ of T_{n+2} which may be in the form 5 or 6, as shown in Figure 36 (in which case we can apply the induction hypothesis) or it may be in one of the other four cases shown in Figure 36.

In Figure 36 we have shown the connectivity patterns that result in T_{n+2} landing in cases 5 or 6. In this figure the rectangular boxes indicate the internal connectivity of T_n, and we have separated these specific cases into three types

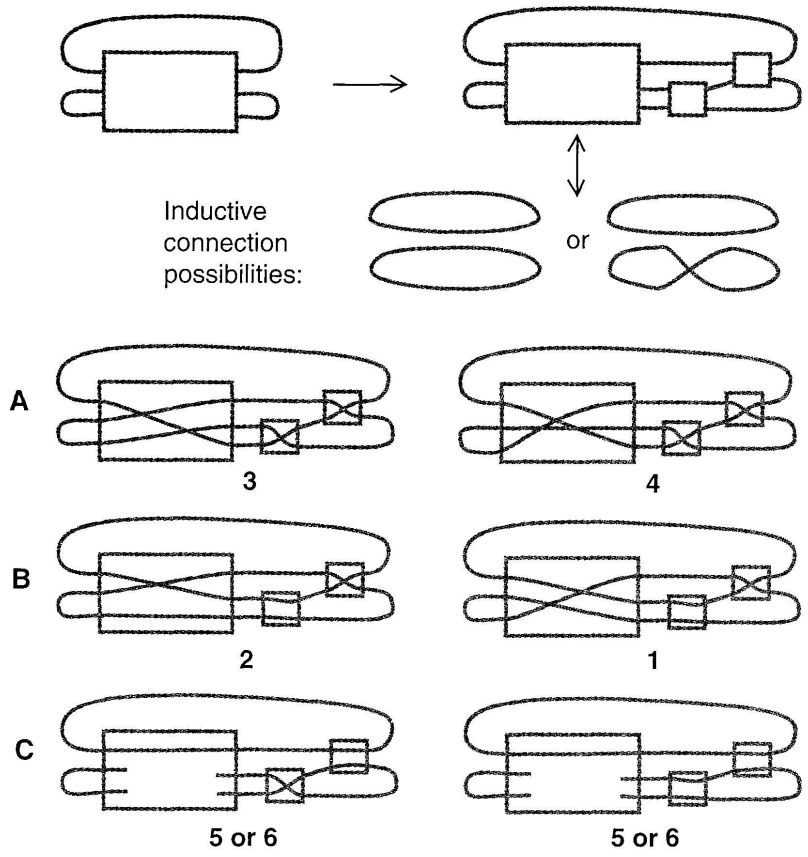


FIGURE 36
Inductive connections

labeled A, B and C (not to be confused with subcases of this proof). In this figure each case is labeled with the type of the predecessor. Thus in the A row one sees the labels 3 and 4 because the boxed patterns are respectively of types 3 and 4. In rows A and B the left hand entries are of type 6 after the addition of the two new terms, and the right hand entries are of type 5. We then check each of these cases to see that the induced value of u in T_{n+2} has the right parity. The calculations can be done by multiplication of generating matrices for continued fractions just using the parity algebra. For example, in Case A of Figure 36 we add two new odd parity terms to T_n in order to obtain T_{n+2}. Thus we multiply the parity matrix for T_n by the product

$$\begin{pmatrix} o & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} o & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e & o \\ o & 1 \end{pmatrix}$$

in order to obtain the parity matrix for T_{n+2}.

In particular, if T_n is in case 3 of Figure 35, then it has fraction parities o/o and o/e and hence parity matrix $\begin{pmatrix} o & e \\ o & o \end{pmatrix}$. Multiplying this by $\begin{pmatrix} e & o \\ o & 1 \end{pmatrix}$,

we obtain

$$\begin{pmatrix} o & e \\ o & o \end{pmatrix} \begin{pmatrix} e & o \\ o & 1 \end{pmatrix} = \begin{pmatrix} e & o \\ o & e \end{pmatrix}.$$

Thus the new u for T_{n+2} is even. Since the connectivity diagram for T_{n+2} in this case, as shown in Figure 36, has compatible standard and palindrome cuts, this result for the parity of u is one step in the verification of the induction hypothesis. Each of the six cases is handled in this same way. We omit the remaining details and assert that the values of u obtained in each case are correct with respect to the connection structure. This completes the proof of Case 2.

Since Cases 1 and 2 encompass all the different possibilities for the standard and palindrome cuts, this completes the proof of the Oriented Schubert Theorem. □

7. Strongly invertible links

An oriented knot or link is invertible if it is oriented isotopic to the link obtained from it by reversing the orientation of each component. We have seen (Lemma 2) that rational knots and links are invertible. A link L of two components is said to be *strongly invertible* if L is ambient isotopic to itself with the orientation of only one component reversed. In Figure 37 we illustrate the link $L = N([[2],[1],[2]])$. This is a strongly invertible link as is apparent by a 180° vertical rotation. This link is well-known as the Whitehead link, a link with linking number zero. Note that since $[[2],[1],[2]]$ has fraction equal to $2 + 1/(1 + 1/2) = 8/3$ this link is non-trivial via the classification of rational knots and links. Note also that $3 \cdot 3 = 1 + 1 \cdot 8$.

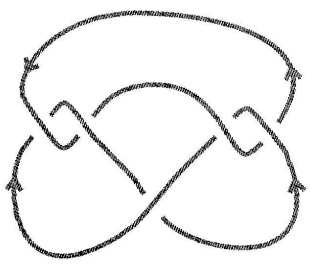


FIGURE 37
The Whitehead link is strongly invertible

In general we have the following

THEOREM 7. *Let $L = N(T)$ be an oriented rational link with associated tangle fraction $F(T) = p/q$ of parity e/o, with p and q relatively prime and $|p| > |q|$. Then L is strongly invertible if and only if $q^2 = 1 + up$ with u an odd integer. It follows that strongly invertible links are all numerators of rational tangles of the form $[[a_1], [a_2], \ldots, [a_k], [\alpha], [a_k], \ldots, [a_2], [a_1]]$ for any integers $a_1, \ldots, a_k, \alpha$.*

Proof. In T the upper two end arcs close to form one component of L and the lower two end arcs close to form the other component of L. Let T' denote the tangle obtained from the oriented tangle T by reversing the orientation of the component containing the lower two arcs and let $N(T') = L'$. (If T'' denotes the tangle obtained from the oriented tangle T by reversing the orientation of the component containing the upper two arcs we have seen that by a vertical 180° rotation the link $N(T')$ is isotopic to the link $N(T'')$. So, for proving Theorem 7 it suffices to consider only the case above.)

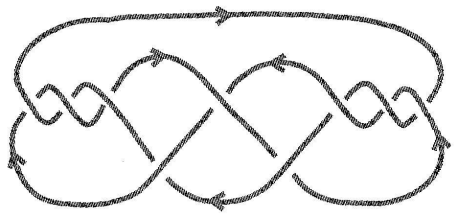
Note that T and T' are incompatible. Thus to apply Theorem 3 we need to perform a bottom twist on T'. Since T and T' have the same fraction p/q, after applying the twist we need to compare the fractions p/q and $p/(p+q)$. Since q is not congruent to $(p+q)$ modulo $2p$, we need to determine when $q(p+q)$ is congruent to 1 modulo $2p$. This will happen exactly when $qp + q^2 = 1 + 2Kp$ for some integer K. The last equation is the same as saying that $q^2 = 1 + up$ with $u = 2K - q$ odd, since q is odd. Now it follows from the Palindrome Theorem for continued fractions that $q^2 = 1 + up$ with u odd and p even if and only if the fraction p/q with $|p| > |q|$ has a palindromic continued fraction expansion with an odd number of terms (the proof is the same in form as the corresponding argument given in the proof of Theorem 5). That is, it has a continued fraction in the form

$$[a_1, a_2, \ldots, a_n, \alpha, a_n, a_{n-1}, \ldots, a_2, a_1].$$

It is then easy to see that the corresponding rational link is ambient isotopic to itself through a vertical 180° rotation. Hence it is strongly invertible. It follows from this that all strongly invertible rational links are ambient isotopic to themselves through a 180° rotation just as in the example of the Whitehead link given above. This completes the proof of the Theorem. □

REMARK 7. Excluding the possibility $T = [\infty]$, as $F(T) = 1/0$ does not have the parity e/o, we may assume $q \neq 0$. And since q is odd (in order

that the rational tangle has two components), the integer $u = 2K - q$ in the equation $q^2 = 1 + up$ cannot be zero. It follows then that the links of the type $N([2n])$, for $n \in \mathbf{Z}$, $n \neq 0$, with tangle fraction $2n/1$ are not invertible (recall the example in Figure 30). Note that, for $n = 0$ we have $T = [0]$ and $F(T) = 0/1$, and in this case Theorem 7 is confirmed, since $1^2 = 1 + u0$, for any u odd. See Figure 38 for another example of a strongly invertible link. In this case the link is $L = N([[3], [1], [1], [1], [3]])$ with $F(L) = 40/11$. Note that $11^2 = 1 + 3 \cdot 40$, fitting the conclusion of Theorem 7.



L = N([[3], [1], [1], [1], [3]])

FIGURE 38
An example of a strongly invertible link

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