Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor: Etingof, Pavel / Strickland, Elisabetta
Kapitel: 1. Lecture 1
DOI: https://doi.org/10.5169/seals-66677

proofs are postponed until Lecture 3). In Lecture 2, we explain the origin of the ring of quasi-invariants in the theory of integrable systems, and introduce some tools from integrable systems, such as the Baker-Akhieser function. Finally, in Lecture 3, we develop the theory of the rational Cherednik algebra, the representation-theoretic techniques due to Opdam and Rouquier, and finish the proofs of the geometric statements from Chapter 1.

1. LECTURE 1

1.1 DEFINITION OF QUASI-INVARIANTS

In this lecture we will define the ring of quasi-invariants Q_m and discuss its main properties.

We will work over the field $\mathbf{C}$ of complex numbers. Let W be a finite Coxeter group, i.e. a finite group generated by reflections. Let us denote by $\mathfrak{h}$ its reflection representation. A typical example is the Weyl group of a semisimple Lie algebra acting on a Cartan subalgebra $\mathfrak{h}$. In the case the Lie algebra is $\mathfrak{sl}(n)$, we have that W is the symmetric group S_n on n letters and $\mathfrak{h}$ is the space of diagonal traceless $n \times n$ matrices.

Let $\Sigma \subset W$ denote the set of reflections. Clearly, W acts on Σ by conjugation. Let $m \colon \Sigma \to \mathbf{Z}_+$ be a function on Σ taking non negative integer values, which is W-invariant. The number of orbits of W on Σ is generally very small. For example, if W is the Weyl group of a simple Lie algebra of ADE type, then W acts transitively on Σ, so m is a constant function.

For each reflection $s \in \Sigma$, choose $\alpha_s \in \mathfrak{h}^* - \{0\}$ so that, for $x \in \mathfrak{h}$, $\alpha_s(sx) = -\alpha_s(x)$ (this means that the hyperplane given by the equation $\alpha_s = 0$ is the reflection hyperplane for s).

DEFINITION 1.1 ([CV1, CV2]). A polynomial $q \in \mathbf{C}[\mathfrak{h}]$ is said to be *m-quasi-invariant* with respect to W if, for any $s \in \Sigma$, the polynomial $q(x) - q(sx)$ is divisible by $\alpha_s(x)^{2m_s+1}$.

We will denote by Q_m the space of m-quasi-invariant polynomials with respect to W.

Notice that every element of $\mathbf{C}[\mathfrak{h}]$ is a 0-quasi-invariant, and that every W-invariant is an m-quasi-invariant for any m. Indeed if $q \in \mathbf{C}[\mathfrak{h}]^W$, then we have $q(x) - q(sx) = 0$ for all $s \in \Sigma$, and 0 is divisible by all powers of $\alpha_s(x)$. Thus in a way, $\mathbf{C}[\mathfrak{h}]^W$ can be viewed as the set of ∞-quasi-invariants.

EXAMPLE 1.2. The group $W = \mathbf{Z}/2$ acts on $\mathfrak{h} = \mathbf{C}$ by $s(v) = -v$. In this case m is a non negative integer and $\Sigma = \{s\}$. So this definition says that q is in Q_m iff $q(x) - q(-x)$ is divisible by x^{2m+1}. It is very easy to write a basis of Q_m. It is given by the polynomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$.

1.2 ELEMENTARY PROPERTIES OF Q_m

Some elementary properties of Q_m are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

1) $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$, $Q_0 = \mathbf{C}[\mathfrak{h}]$, $Q_m \subset Q_{m'}$ *if* $m \geq m'$, $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$.
2) Q_m *is a graded subalgebra of* $\mathbf{C}[\mathfrak{h}]$.
3) *The fraction field of* Q_m *is equal to* $\mathbf{C}(\mathfrak{h})$.
4) Q_m *is a finite* $\mathbf{C}[\mathfrak{h}]^W$*-module and a finitely generated algebra.* $\mathbf{C}[\mathfrak{h}]$ *is a finite* Q_m*-module.*

Proof. 1) is immediate and has already been mentioned in 1.1.

2) Clearly Q_m is closed under addition. Let $p, q \in Q_m$. Let $s \in \Sigma$. Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both $p(x) - p(sx)$ and $q(x) - q(sx)$ are divisible by $\alpha_s^{2m_s+1}$, we deduce that $p(x)q(x) - p(sx)q(sx)$ is also divisible by $\alpha_s^{2m_s+1}$, proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$ for each $s \in \Sigma$, hence $\delta_{2m+1} \in Q_m$. Take $f(x) \in \mathbf{C}[\mathfrak{h}]$. We claim that $f(x)\delta_{2m+1}(x) \in Q_m$. As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition $\delta_{2m+1}(x)$ is divisible by $\alpha_s(x)^{2m_s+1}$ for all $s \in \Sigma$. This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that $\mathbf{C}[\mathfrak{h}]^W$ is a finitely generated algebra over $\mathbf{C}$ and $\mathbf{C}[\mathfrak{h}]$ is a finite $\mathbf{C}[\mathfrak{h}]^W$-module and hence a finite Q_m-module, proving the second part of 4).

Now $Q_m \subset \mathbf{C}[\mathfrak{h}]$ is a submodule of the finite module $\mathbf{C}[\mathfrak{h}]$ over the Noetherian ring $\mathbf{C}[\mathfrak{h}]^W$. Hence it is finite. This immediately implies that Q_m is a finitely generated algebra over $\mathbf{C}$. $\square$

REMARK. In fact, since W is a finite Coxeter group, a celebrated result of Chevalley says that the algebra $\mathbf{C}[\mathfrak{h}]^W$ is not only a finitely generated $\mathbf{C}$-algebra but actually a free (=polynomial) algebra. Namely, it is of the form $\mathbf{C}[q_1, \dots, q_n]$, where the q_i are homogeneous polynomials of some degrees d_i. Furthermore, if we denote by H the subspace of $\mathbf{C}[\mathfrak{h}]$ of harmonic polynomials, i.e. of polynomials killed by W-invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \to \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of $\mathbf{C}[\mathfrak{h}]^W$- and of W-modules. In particular, $\mathbf{C}[\mathfrak{h}]$ is a free $\mathbf{C}[\mathfrak{h}]^W$-module of rank $|W|$.

1.3 THE VARIETY X_m AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety $X_m = \operatorname{Spec}(Q_m)$. The inclusion $Q_m \subset \mathbf{C}[\mathfrak{h}]$ induces a morphism

$$\pi \colon \mathfrak{h} \to X_m ,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that X_m is singular for all $m \neq 0$.)

In fact, not only is π birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]). *π is a bijection.*

Proof. By the above remarks, we only have to show that π is injective. In order to achieve this, we need to prove that quasi-invariants separate points of $\mathfrak{h}$, i.e. that if $z, y \in \mathfrak{h}$ and $z \neq y$, then there exists $p \in Q_m$ such that $p(z) \neq p(y)$. This is obtained in the following way. Let $W_z \subset W$ be the stabilizer of z and choose $f \in \mathbf{C}[\mathfrak{h}]$ such that $f(z) \neq 0$, $f(y) = 0$. Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx) .$$

We claim that $p(x) \in Q_m$. Indeed, let $s \in \Sigma$ and assume that $s(z) \neq z$.

We have by definition $p(x) = \alpha_s(x)^{2m_s+1}\tilde{p}(x)$, with $\tilde{p}(x)$ a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1}\tilde{p}(x) - \alpha_s(sx)^{2m_s+1}\tilde{p}(sx) = \alpha_s(x)^{2m_s+1}(\tilde{p}(x) + \tilde{p}(sx)) .$$

If on the other hand, $sz = z$, i.e. $s \in W_z$, then s preserves the set $W \setminus W_z$, and hence preserves $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$ (as it acts by -1 on the products $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$ and $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$). Since $\prod_{w \in W_z} f(wx)$ is

W_z-invariant, we deduce that $p(x)-p(sx)=0$, so that in this case $p(x)-p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z)\neq 0$. Indeed, for a reflection s, α_s vanishes exactly on the fixed points of s, so that $\prod_{s\in\Sigma, sz\neq z}\alpha_s(z)^{2m_s+1}\neq 0$. Also for all $w\in W_z$ $f(wz)=f(z)\neq 0$. On the other hand, it is clear that $p(y)=0$. $\square$

EXAMPLE 1.5. Take $W=\mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i}\mid i\geq 0\}\cup\{x^{2i+1}\mid i\geq m\}$. From this we deduce that setting $z=x^2$ and $y=x^{2m+1}$, $Q_m=\mathbf{C}[y,z]/(y^2-z^{2m+1})=\mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2=z^{2m+1}$. The map $\pi\colon\mathbf{C}\to K$ is given by $\pi(t)=(t^{2m+1},t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over $\mathbf{C}$ and $A=\mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1,\dots,f_n\in\mathbf{C}[X]$ which are algebraically independent over $\mathbf{C}$ and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist $f_1,\dots,f_n$ as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1,\dots,f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any $f_1,\dots,f_n$ which are algebraically independent over $\mathbf{C}$ and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$, we have that A is a locally free $\mathbf{C}[f_1,\dots,f_n]$-module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). *Q_m is Cohen-Macaulay.*

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). *Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module.*

We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the $\mathbf{C}[\mathfrak{h}]^W$-module Q_m can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category $\mathcal{O}$. On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra $\mathbf{C}[\mathfrak{h}]^W$.

1.5 THE POINCARÉ SERIES OF Q_m

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r ,$$

where $Q_m[r]$ denotes the graded component of Q_m of degree r. For every irreducible representation $\tau \in \widehat{W}$, define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}][r]) t^r .$$

Consider the element in the group ring $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s(1 - s) .$$

The W-invariance of m implies that μ_m lies in the center of $\mathbf{Z}[W]$. Hence it is clear that μ_m acts as a scalar, $\xi_m(\tau)$, on τ. Let d_τ be the degree of τ.

LEMMA 1.9. *The scalar $\xi_m(\tau)$ is an integer.*

Proof. $\mathbf{Z}[W]$ and hence also its center, is a finite $\mathbf{Z}$-module. This clearly implies that $\xi_m(\tau)$ is an algebraic integer. Thus to prove that $\xi_m(\tau)$ is an integer, it suffices to see that $\xi_m(\tau)$ is a rational number. Let $d_{\tau,s}$ be the dimension of the space of s-invariants in τ. Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2m_s(d_\tau - d_{\tau,s}) ,$$

which gives the rationality of $\xi_m(\tau)$. □

THEOREM 1.10. *One has*

$$h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} \chi_\tau(t)\,. \tag{1}$$

REMARK. This theorem was proved in [FeV] modulo Theorem 1.7 (conjectured in [FV]) using the so-called Matsuo-Cherednik correspondence (see [FeV] for details). Thus, Theorem 1.10 follows from [FeV] and [EG2]. Another proof of this theorem is given in [BEG]; this is the proof we will discuss below (in Lecture 3).

EXAMPLE 1.11. If $m = 0$, since $Q_0 = \mathbf{C}[\mathfrak{h}]$, the theorem says that

$$h_{Q_0}(t) = \frac{1}{(1-t)^n} = \sum_{\tau \in \widehat{W}} d_\tau \chi_\tau(t)\,.$$

Indeed, as a W-module one has

$$\mathbf{C}[\mathfrak{h}] = \oplus_\tau \tau \otimes \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}])\,.$$

EXAMPLE 1.12. If $W = \mathbf{Z}/2$, then $\widehat{W} = \{+,-\}$, where $+$ (respectively $-$) denotes the trivial (respectively the sign) representation. One has

$$\mathbf{C}[x] = \mathbf{C}[x^2] \oplus \mathbf{C}[x^2]x\,,$$

where $\mathbf{C}[x^2] = \mathbf{C}[x]^W$ and $\mathbf{C}[x^2]x$ is the isotypic component of the sign representation. Thus

$$\chi_+(t) = \frac{1}{1-t^2}\,, \quad \chi_-(t) = \frac{t}{1-t^2}\,,$$

$\mu_m = m(1-s)$. Thus $\xi_m(+) = 0$, $\xi_m(-) = 2m$. We deduce that

$$h_{Q_m}(t) = \frac{1}{1-t^2} + \frac{t^{2m+1}}{1-t^2}\,,$$

as we already know.

Recall now that as a graded W-module $\mathbf{C}[\mathfrak{h}]$ is isomorphic to $\mathbf{C}[\mathfrak{h}]^W \otimes H$, H being the space of harmonic polynomials. We deduce that the τ-isotypic component in $\mathbf{C}[\mathfrak{h}]$ is isomorphic to $\mathbf{C}[\mathfrak{h}]^W \otimes H_\tau$.

Set $K_\tau(t) = \sum_{r\geq 0} \dim \operatorname{Hom}_W(\tau, H[r])t^r$. This is a polynomial, called the Kostka polynomial relative to τ. We deduce that

$$\chi_\tau(t) = \frac{K_\tau(t)}{\prod_{i=1}^n (1-t^{d_i})}. \tag{2}$$

Also, if $\tau' = \tau \otimes \varepsilon$, ε being the sign representation, one has

$$K_{\tau'}(t) = K_\tau(t^{-1})t^{|\Sigma|}.$$

Set now

$$P_m(t) = \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\tau)} K_\tau(t).$$

We have

PROPOSITION 1.13 ([FeV]).

$$h_{Q_m}(t) = \frac{P_m(t)}{\prod_{i=1}^n (1-t^{d_i})}.$$

Furthermore $P_m(t) = t^{\xi_m(\varepsilon)+|\Sigma|} P_m(t^{-1})$.

Proof. Substituting the expression (2) for $\chi_\tau(t)$ in (1.10) and using the definition of $P_m(t)$, we get

$$h_{Q_m}(t) = \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\tau)} \frac{K_\tau(t)}{\prod_{i=1}^n (1-t^{d_i})} = \frac{P_m(t)}{\prod_{i=1}^n (1-t^{d_i})},$$

as desired.

Now notice that

$$\xi_m(\tau) + \xi_m(\tau') = \sum_{s\in\Sigma} 2m_s = \xi_m(\varepsilon).$$

Using this we get

$$\begin{aligned} t^{\xi_m(\varepsilon)+|\Sigma|} P_m(t^{-1}) &= \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\varepsilon)-\xi_m(\tau)} t^{|\Sigma|} K_\tau(t^{-1}) \\ &= \sum_{\tau'\in\widehat{W}} d_{\tau'} t^{\xi_m(\tau')} K_{\tau'}(t) = P_m(t), \end{aligned}$$

as desired. $\square$

From this we deduce

THEOREM 1.14 ([EG2, BEG, FeV], conjectured in [FV]). *The ring Q_m of m-quasi-invariants is Gorenstein.*

Proof. By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain A is Gorenstein iff its Poincaré series is a rational function $h(t)$ satisfying the equation $h(t^{-1}) = (-1)^n t^l h(t)$, where l is an integer and n is the dimension of the spectrum of A. Thus the result follows immediately from Proposition 1.13. □

1.6 THE RING OF DIFFERENTIAL OPERATORS ON X_m

Finally, let us introduce the ring $\mathcal{D}(X_m)$ of differential operators on X_m, that is the ring of differential operators with coefficients in $\mathbf{C}(\mathfrak{h})$ mapping Q_m to Q_m. It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

THEOREM 1.15 ([BEG]). *$\mathcal{D}(X_m)$ is a simple algebra.*

REMARK 1.16. a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

2. LECTURE 2

We will now see how the ring Q_m appears in the theory of completely integrable systems.

2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space X (a smooth manifold). Then the phase space of this system is T^*X, the cotangent bundle on X. The space T^*X is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on T^*X. A point of T^*X is a pair (x,p), where $x \in X$ is the position and $p \in T_x^*X$ is the momentum. Such pairs are