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proofs are postponed until Lecture 3). In Lecture 2, we explain the origin of the ring of quasi-invariants in the theory of integrable systems, and introduce some tools from integrable systems, such as the Baker-Akhieser function. Finally, in Lecture 3, we develop the theory of the rational Cherednik algebra, the representation-theoretic techniques due to Opdam and Rouquier, and finish the proofs of the geometric statements from Chapter 1.

## 1. LECTURE 1

### 1.1 DEFINITION OF QUASI-INVARIANTS

In this lecture we will define the ring of quasi-invariants  $Q_m$  and discuss its main properties.

We will work over the field  $\mathbf{C}$  of complex numbers. Let  $W$  be a finite Coxeter group, i.e. a finite group generated by reflections. Let us denote by  $\mathfrak{h}$  its reflection representation. A typical example is the Weyl group of a semisimple Lie algebra acting on a Cartan subalgebra  $\mathfrak{h}$ . In the case the Lie algebra is  $\mathfrak{sl}(n)$ , we have that  $W$  is the symmetric group  $S_n$  on  $n$  letters and  $\mathfrak{h}$  is the space of diagonal traceless  $n \times n$  matrices.

Let  $\Sigma \subset W$  denote the set of reflections. Clearly,  $W$  acts on  $\Sigma$  by conjugation. Let  $m: \Sigma \rightarrow \mathbf{Z}_+$  be a function on  $\Sigma$  taking non negative integer values, which is  $W$ -invariant. The number of orbits of  $W$  on  $\Sigma$  is generally very small. For example, if  $W$  is the Weyl group of a simple Lie algebra of ADE type, then  $W$  acts transitively on  $\Sigma$ , so  $m$  is a constant function.

For each reflection  $s \in \Sigma$ , choose  $\alpha_s \in \mathfrak{h}^* - \{0\}$  so that, for  $x \in \mathfrak{h}$ ,  $\alpha_s(sx) = -\alpha_s(x)$  (this means that the hyperplane given by the equation  $\alpha_s = 0$  is the reflection hyperplane for  $s$ ).

**DEFINITION 1.1** ([CV1, CV2]). A polynomial  $q \in \mathbf{C}[\mathfrak{h}]$  is said to be  *$m$ -quasi-invariant* with respect to  $W$  if, for any  $s \in \Sigma$ , the polynomial  $q(x) - q(sx)$  is divisible by  $\alpha_s(x)^{2m_s+1}$ .

We will denote by  $Q_m$  the space of  $m$ -quasi-invariant polynomials with respect to  $W$ .

Notice that every element of  $\mathbf{C}[\mathfrak{h}]$  is a 0-quasi-invariant, and that every  $W$ -invariant is an  $m$ -quasi-invariant for any  $m$ . Indeed if  $q \in \mathbf{C}[\mathfrak{h}]^W$ , then we have  $q(x) - q(sx) = 0$  for all  $s \in \Sigma$ , and 0 is divisible by all powers of  $\alpha_s(x)$ . Thus in a way,  $\mathbf{C}[\mathfrak{h}]^W$  can be viewed as the set of  $\infty$ -quasi-invariants.

EXAMPLE 1.2. The group  $W = \mathbf{Z}/2$  acts on  $\mathfrak{h} = \mathbf{C}$  by  $s(v) = -v$ . In this case  $m$  is a non negative integer and  $\Sigma = \{s\}$ . So this definition says that  $q$  is in  $Q_m$  iff  $q(x) - q(-x)$  is divisible by  $x^{2m+1}$ . It is very easy to write a basis of  $Q_m$ . It is given by the polynomials  $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$ .

## 1.2 ELEMENTARY PROPERTIES OF $Q_m$

Some elementary properties of  $Q_m$  are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

- 1)  $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$ ,  $Q_0 = \mathbf{C}[\mathfrak{h}]$ ,  $Q_m \subset Q_{m'}$  if  $m \geq m'$ ,  $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$ .
- 2)  $Q_m$  is a graded subalgebra of  $\mathbf{C}[\mathfrak{h}]$ .
- 3) The fraction field of  $Q_m$  is equal to  $\mathbf{C}(\mathfrak{h})$ .
- 4)  $Q_m$  is a finite  $\mathbf{C}[\mathfrak{h}]^W$ -module and a finitely generated algebra.  $\mathbf{C}[\mathfrak{h}]$  is a finite  $Q_m$ -module.

*Proof.* 1) is immediate and has already been mentioned in 1.1.

2) Clearly  $Q_m$  is closed under addition. Let  $p, q \in Q_m$ . Let  $s \in \Sigma$ . Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both  $p(x) - p(sx)$  and  $q(x) - q(sx)$  are divisible by  $\alpha_s^{2m_s+1}$ , we deduce that  $p(x)q(x) - p(sx)q(sx)$  is also divisible by  $\alpha_s^{2m_s+1}$ , proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has  $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$  for each  $s \in \Sigma$ , hence  $\delta_{2m+1} \in Q_m$ . Take  $f(x) \in \mathbf{C}[\mathfrak{h}]$ . We claim that  $f(x)\delta_{2m+1}(x) \in Q_m$ . As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition  $\delta_{2m+1}(x)$  is divisible by  $\alpha_s(x)^{2m_s+1}$  for all  $s \in \Sigma$ . This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that  $\mathbf{C}[\mathfrak{h}]^W$  is a finitely generated algebra over  $\mathbf{C}$  and  $\mathbf{C}[\mathfrak{h}]$  is a finite  $\mathbf{C}[\mathfrak{h}]^W$ -module and hence a finite  $Q_m$ -module, proving the second part of 4).

Now  $Q_m \subset \mathbf{C}[\mathfrak{h}]$  is a submodule of the finite module  $\mathbf{C}[\mathfrak{h}]$  over the Noetherian ring  $\mathbf{C}[\mathfrak{h}]^W$ . Hence it is finite. This immediately implies that  $Q_m$  is a finitely generated algebra over  $\mathbf{C}$ .  $\square$

REMARK. In fact, since  $W$  is a finite Coxeter group, a celebrated result of Chevalley says that the algebra  $\mathbf{C}[\mathfrak{h}]^W$  is not only a finitely generated  $\mathbf{C}$ -algebra but actually a free (=polynomial) algebra. Namely, it is of the form  $\mathbf{C}[q_1, \dots, q_n]$ , where the  $q_i$  are homogeneous polynomials of some degrees  $d_i$ . Furthermore, if we denote by  $H$  the subspace of  $\mathbf{C}[\mathfrak{h}]$  of harmonic polynomials, i.e. of polynomials killed by  $W$ -invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \rightarrow \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of  $\mathbf{C}[\mathfrak{h}]^W$ - and of  $W$ -modules. In particular,  $\mathbf{C}[\mathfrak{h}]$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module of rank  $|W|$ .

### 1.3 THE VARIETY $X_m$ AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety  $X_m = \text{Spec}(Q_m)$ . The inclusion  $Q_m \subset \mathbf{C}[\mathfrak{h}]$  induces a morphism

$$\pi: \mathfrak{h} \rightarrow X_m,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that  $X_m$  is singular for all  $m \neq 0$ .)

In fact, not only is  $\pi$  birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]).  *$\pi$  is a bijection.*

*Proof.* By the above remarks, we only have to show that  $\pi$  is injective. In order to achieve this, we need to prove that quasi-invariants separate points of  $\mathfrak{h}$ , i.e. that if  $z, y \in \mathfrak{h}$  and  $z \neq y$ , then there exists  $p \in Q_m$  such that  $p(z) \neq p(y)$ . This is obtained in the following way. Let  $W_z \subset W$  be the stabilizer of  $z$  and choose  $f \in \mathbf{C}[\mathfrak{h}]$  such that  $f(z) \neq 0$ ,  $f(y) = 0$ . Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx).$$

We claim that  $p(x) \in Q_m$ . Indeed, let  $s \in \Sigma$  and assume that  $s(z) \neq z$ .

We have by definition  $p(x) = \alpha_s(x)^{2m_s+1} \tilde{p}(x)$ , with  $\tilde{p}(x)$  a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1} \tilde{p}(x) - \alpha_s(sx)^{2m_s+1} \tilde{p}(sx) = \alpha_s(x)^{2m_s+1} (\tilde{p}(x) + \tilde{p}(sx)).$$

If on the other hand,  $sz = z$ , i.e.  $s \in W_z$ , then  $s$  preserves the set  $W \setminus W_z$ , and hence preserves  $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$  (as it acts by  $-1$  on the products  $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$  and  $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$ ). Since  $\prod_{w \in W_z} f(wx)$  is

$W_z$ -invariant, we deduce that  $p(x) - p(sx) = 0$ , so that in this case  $p(x) - p(sx)$  also is divisible by  $\alpha_s(x)^{2m_s+1}$ .

To conclude, notice that  $p(z) \neq 0$ . Indeed, for a reflection  $s$ ,  $\alpha_s$  vanishes exactly on the fixed points of  $s$ , so that  $\prod_{s \in \Sigma, sz \neq z} \alpha_s(z)^{2m_s+1} \neq 0$ . Also for all  $w \in W_z$   $f(wz) = f(z) \neq 0$ . On the other hand, it is clear that  $p(y) = 0$ .  $\square$

EXAMPLE 1.5. Take  $W = \mathbf{Z}/2$ . As we have already seen,  $Q_m$  has a basis given by the monomials  $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$ . From this we deduce that setting  $z = x^2$  and  $y = x^{2m+1}$ ,  $Q_m = \mathbf{C}[y, z]/(y^2 - z^{2m+1}) = \mathbf{C}[K]$ , where  $K$  is the plane curve with a cusp at the origin, given by the equation  $y^2 = z^{2m+1}$ . The map  $\pi: \mathbf{C} \rightarrow K$  is given by  $\pi(t) = (t^{2m+1}, t^2)$ , which is clearly bijective.

#### 1.4 FURTHER PROPERTIES OF $X_m$

Let us get to some deeper properties of quasi-invariants. Let  $X$  be an irreducible affine variety over  $\mathbf{C}$  and  $A = \mathbf{C}[X]$ . Recall that, by the Noether Normalization Lemma, there exist  $f_1, \dots, f_n \in \mathbf{C}[X]$  which are algebraically independent over  $\mathbf{C}$  and such that  $\mathbf{C}[X]$  is a finite module over the polynomial ring  $\mathbf{C}[f_1, \dots, f_n]$ . This means that we have a finite morphism of  $X$  onto an affine space.

DEFINITION 1.6.  $A$  (and  $X$ ) is said to be *Cohen-Macaulay* if there exist  $f_1, \dots, f_n$  as above, with the property that  $\mathbf{C}[X]$  is a locally free module over  $\mathbf{C}[f_1, \dots, f_n]$ . (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that  $A$  is a free module.)

REMARK. If  $A$  is Cohen-Macaulay, then for any  $f_1, \dots, f_n$  which are algebraically independent over  $\mathbf{C}$  and such that  $A$  is a finite module over the polynomial ring  $\mathbf{C}[f_1, \dots, f_n]$ , we have that  $A$  is a locally free  $\mathbf{C}[f_1, \dots, f_n]$ -module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]).  $Q_m$  is Cohen-Macaulay.

Notice that, using Chevalley's result that  $\mathbf{C}[\mathfrak{h}]^W$  is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]).  $Q_m$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module.

We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the  $\mathbf{C}[\hbar]^W$ -module  $Q_m$  can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category  $\mathcal{O}$ . On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra  $\mathbf{C}[\hbar]^W$ .

### 1.5 THE POINCARÉ SERIES OF $Q_m$

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r,$$

where  $Q_m[r]$  denotes the graded component of  $Q_m$  of degree  $r$ . For every irreducible representation  $\tau \in \widehat{W}$ , define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \operatorname{Hom}_W(\tau, \mathbf{C}[\hbar][r]) t^r.$$

Consider the element in the group ring  $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s).$$

The  $W$ -invariance of  $m$  implies that  $\mu_m$  lies in the center of  $\mathbf{Z}[W]$ . Hence it is clear that  $\mu_m$  acts as a scalar,  $\xi_m(\tau)$ , on  $\tau$ . Let  $d_\tau$  be the degree of  $\tau$ .

LEMMA 1.9. *The scalar  $\xi_m(\tau)$  is an integer.*

*Proof.*  $\mathbf{Z}[W]$  and hence also its center, is a finite  $\mathbf{Z}$ -module. This clearly implies that  $\xi_m(\tau)$  is an algebraic integer. Thus to prove that  $\xi_m(\tau)$  is an integer, it suffices to see that  $\xi_m(\tau)$  is a rational number. Let  $d_{\tau,s}$  be the dimension of the space of  $s$ -invariants in  $\tau$ . Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2m_s (d_\tau - d_{\tau,s}),$$

which gives the rationality of  $\xi_m(\tau)$ .  $\square$

THEOREM 1.10. *One has*

$$(1) \quad h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_{\tau} t^{\xi_m(\tau)} \chi_{\tau}(t).$$

REMARK. This theorem was proved in [FeV] modulo Theorem 1.7 (conjectured in [FV]) using the so-called Matsuo-Cherednik correspondence (see [FeV] for details). Thus, Theorem 1.10 follows from [FeV] and [EG2]. Another proof of this theorem is given in [BEG]; this is the proof we will discuss below (in Lecture 3).

EXAMPLE 1.11. If  $m = 0$ , since  $Q_0 = \mathbf{C}[\mathfrak{h}]$ , the theorem says that

$$h_{Q_0}(t) = \frac{1}{(1-t)^n} = \sum_{\tau \in \widehat{W}} d_{\tau} \chi_{\tau}(t).$$

Indeed, as a  $W$ -module one has

$$\mathbf{C}[\mathfrak{h}] = \bigoplus_{\tau} \tau \otimes \text{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}]).$$

EXAMPLE 1.12. If  $W = \mathbf{Z}/2$ , then  $\widehat{W} = \{+, -\}$ , where  $+$  (respectively  $-$ ) denotes the trivial (respectively the sign) representation. One has

$$\mathbf{C}[x] = \mathbf{C}[x^2] \oplus \mathbf{C}[x^2]x,$$

where  $\mathbf{C}[x^2] = \mathbf{C}[x]^W$  and  $\mathbf{C}[x^2]x$  is the isotypic component of the sign representation. Thus

$$\chi_+(t) = \frac{1}{1-t^2}, \quad \chi_-(t) = \frac{t}{1-t^2},$$

$\mu_m = m(1-s)$ . Thus  $\xi_m(+) = 0$ ,  $\xi_m(-) = 2m$ . We deduce that

$$h_{Q_m}(t) = \frac{1}{1-t^2} + \frac{t^{2m+1}}{1-t^2},$$

as we already know.

Recall now that as a graded  $W$ -module  $\mathbf{C}[\mathfrak{h}]$  is isomorphic to  $\mathbf{C}[\mathfrak{h}]^W \otimes H$ ,  $H$  being the space of harmonic polynomials. We deduce that the  $\tau$ -isotypic component in  $\mathbf{C}[\mathfrak{h}]$  is isomorphic to  $\mathbf{C}[\mathfrak{h}]^W \otimes H_{\tau}$ .

Set  $K_\tau(t) = \sum_{r \geq 0} \dim \operatorname{Hom}_W(\tau, H[r])t^r$ . This is a polynomial, called the Kostka polynomial relative to  $\tau$ . We deduce that

$$(2) \quad \chi_\tau(t) = \frac{K_\tau(t)}{\prod_{i=1}^n (1 - t^{d_i})}.$$

Also, if  $\tau' = \tau \otimes \varepsilon$ ,  $\varepsilon$  being the sign representation, one has

$$K_{\tau'}(t) = K_\tau(t^{-1})t^{|\Sigma|}.$$

Set now

$$P_m(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} K_\tau(t).$$

We have

PROPOSITION 1.13 ([FeV]).

$$h_{Q_m}(t) = \frac{P_m(t)}{\prod_{i=1}^n (1 - t^{d_i})}.$$

Furthermore  $P_m(t) = t^{\xi_m(\varepsilon) + |\Sigma|} P_m(t^{-1})$ .

*Proof.* Substituting the expression (2) for  $\chi_\tau(t)$  in (1.10) and using the definition of  $P_m(t)$ , we get

$$h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} \frac{K_\tau(t)}{\prod_{i=1}^n (1 - t^{d_i})} = \frac{P_m(t)}{\prod_{i=1}^n (1 - t^{d_i})},$$

as desired.

Now notice that

$$\xi_m(\tau) + \xi_m(\tau') = \sum_{s \in \Sigma} 2m_s = \xi_m(\varepsilon).$$

Using this we get

$$\begin{aligned} t^{\xi_m(\varepsilon) + |\Sigma|} P_m(t^{-1}) &= \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\varepsilon) - \xi_m(\tau)} t^{|\Sigma|} K_\tau(t^{-1}) \\ &= \sum_{\tau' \in \widehat{W}} d_{\tau'} t^{\xi_m(\tau')} K_{\tau'}(t) = P_m(t), \end{aligned}$$

as desired.  $\square$



From this we deduce

THEOREM 1.14 ([EG2, BEG, FeV], conjectured in [FV]). *The ring  $Q_m$  of  $m$ -quasi-invariants is Gorenstein.*

*Proof.* By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain  $A$  is Gorenstein iff its Poincaré series is a rational function  $h(t)$  satisfying the equation  $h(t^{-1}) = (-1)^n t^l h(t)$ , where  $l$  is an integer and  $n$  is the dimension of the spectrum of  $A$ . Thus the result follows immediately from Proposition 1.13.  $\square$

## 1.6 THE RING OF DIFFERENTIAL OPERATORS ON $X_m$

Finally, let us introduce the ring  $\mathcal{D}(X_m)$  of differential operators on  $X_m$ , that is the ring of differential operators with coefficients in  $\mathbb{C}(\mathfrak{h})$  mapping  $Q_m$  to  $Q_m$ . It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

THEOREM 1.15 ([BEG]).  *$\mathcal{D}(X_m)$  is a simple algebra.*

REMARK 1.16. a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

## 2. LECTURE 2

We will now see how the ring  $Q_m$  appears in the theory of completely integrable systems.

### 2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space  $X$  (a smooth manifold). Then the phase space of this system is  $T^*X$ , the cotangent bundle on  $X$ . The space  $T^*X$  is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on  $T^*X$ . A point of  $T^*X$  is a pair  $(x, p)$ , where  $x \in X$  is the position and  $p \in T_x^*X$  is the momentum. Such pairs are