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All of these constructions can be made equivariant in a rather obvious way: Thus if  $G$  is another Lie group and  $P$  is a  $G$ -invariant principal  $K$ -bundle, any  $\varrho \in \text{Hom}(\pi_1(K), \text{U}(1))$  defines a  $G$ -equivariant bundle gerbe  $(P, L, t)$  (with flat connection) over  $M$ . If  $\varrho$  is in the image of  $\mu \in (\mathfrak{k}^*)^K$ , there is a  $G$ -equivariant pseudo-line bundle for this gerbe. Furthermore any choice of  $G$ -equivariant principal connection on  $P$  defines a  $G$ -equivariant pseudo-line bundle connection, with equivariant error 2-form  $\pi^* \omega_G = \langle \mu, F_G^\theta \rangle$  where  $F_G^\theta \in \Omega_G^2(P, \mathfrak{k})$  is the equivariant curvature.

#### 4. GLUING DATA

In this Section we describe a procedure for gluing a collection of bundle gerbes  $(X_i, L_i, t_i)$  on open subsets  $V_i \subset M$ , with pseudo-line bundles of their quotients on overlaps<sup>2)</sup>. We begin with the somewhat simpler case that the surjective submersions  $X_i \rightarrow V_i$  are obtained by restricting a surjective submersion  $X \rightarrow M$ , and later reduce the general case to this special case.

Thus, let  $\pi: X \rightarrow M$  be a surjective submersion and let  $V_i$ ,  $i = 0, \dots, d$  an open cover of  $M$ . Let  $X_i = X|_{V_i}$ , and more generally  $X_I = X|_{V_I}$  where  $V_I$  is the intersection of all  $V_i$  with  $i \in I$ .

Suppose we are given bundle gerbes  $(X_i, L_i, t_i)$  over  $V_i$  and pseudo-line bundles  $(E_{ij}, s_{ij})$  for the quotients  $(X_{ij}, L_j L_i^{-1}, t_j t_i^{-1})$  over  $V_i \cap V_j$ , where  $E_{ij} = E_{ji}^{-1}$  and  $s_{ij} = s_{ji}^{-1}$ . Note that  $E_{ij} E_{jk} E_{ki}$  is a pseudo-line bundle for the trivial gerbe, hence is a pull-back  $\pi^* F_{ijk}$  of a line bundle  $F_{ijk} \rightarrow M$ , and we will also require a unitary section  $u_{ijk}$  of that line bundle. Under suitable conditions the data  $(E_{ij}, s_{ij})$  and  $u_{ijk}$  can be used to 'glue' the gerbes  $(X_i, L_i, t_i)$ . The glued gerbe will be defined over the disjoint union  $\coprod_{i=1}^d X_i$ . We have

$$\begin{aligned} \left( \coprod_{i=1}^d X_i \right)^{[2]} &= \coprod_{ij} X_i \times_M X_j \\ \left( \coprod_{i=1}^d X_i \right)^{[3]} &= \coprod_{ijk} X_i \times_M X_j \times_M X_k \\ &\dots \end{aligned}$$

Hence, the glued gerbe will be of the form  $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$  where  $L_{ij}$  are line bundles over  $X_i \times_M X_j$  and  $t_{ijk}$  unitary sections of a line bundle  $(\delta L)_{ijk}$

<sup>2)</sup> See Stevenson [29] for similar gluing constructions.

over  $\coprod_{ijk} X_i \times_M X_j \times_M X_k$ . We will define  $L_{ij}$  by tensoring  $L_i \rightarrow X^{[2]}$  (restricted to  $X_i \times_M X_j$ ) with the pull-back of  $E_{ij}$  under the map  $\partial_1: X_i \times_M X_j \rightarrow X_{ij}$ .

**PROPOSITION 4.1.** *Suppose the sections  $u_{ijk}$  satisfy the cocycle condition  $u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1} = 1$ , and the sections  $s_{ij}$  satisfy a cocycle condition  $s_{ij}s_{jk}s_{ki} = 1$ . Then there is a well-defined gerbe  $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$  over  $M$ , where  $L_{ij} \rightarrow X_i \times_M X_j$  is the line bundle*

$$L_{ij} = L_j \otimes \partial_1^* E_{ij}$$

and  $t_{ijk}$  is a section of  $(\delta L)_{ijk} \rightarrow X_i \times_M X_j \times_M X_k$  given by

$$(4.1) \quad t_{ijk} = t_k \otimes \partial_2^* s_{kj} \otimes \partial_2^* \partial_1^* \pi^* u_{ijk}.$$

*Proof.* A short calculation gives

$$(\delta L)_{ijk} = (\delta L_k) \otimes \partial_2^* (L_j L_k^{-1} \delta E_{kj}^{-1}) \otimes \partial_2^* \partial_1^* \pi^* F_{ijk},$$

showing that  $t_{ijk}$  is a well-defined section of  $(\delta L)_{ijk}$ . One finds furthermore

$$\begin{aligned} (\delta t)_{ijkl} &= (\delta t_l) \otimes \partial_3^* (t_l t_k^{-1} \delta s_{kl}^{-1} \otimes \partial_2^* (s_{lj} s_{jk} s_{kl} \otimes \partial_1^* \pi^* (u_{jkl} u_{ikl}^{-1} u_{ijl} u_{ijk}^{-1}))) \\ &= \partial_3^* \partial_2^* (s_{lj} s_{jk} s_{kl} \otimes \partial_1^* \pi^* (u_{jkl} u_{ikl}^{-1} u_{ijl} u_{ijk}^{-1})) \end{aligned}$$

which equals 1 under the given assumptions on  $u$  and  $s$ .

The gluing construction described in this Proposition is particularly natural for Chatterjee-Hitchin gerbes: Suppose  $\mathcal{U}$  is an open cover of  $M$ , and  $X = \coprod_{U \in \mathcal{U}} U$ . For any decomposition  $\mathcal{U} = \coprod_{i=1}^d \mathcal{U}_i$  let  $V_i = \cup_{U \in \mathcal{U}_i} U$ , and  $X_i = \coprod_{U \in \mathcal{U}_i} U$ . Note that in this case,

$$\coprod_i X_i = X.$$

Suppose  $(L_i, t_i)$  are Chatterjee-Hitchin gerbes for the cover  $\mathcal{U}_i$  of  $V_i$ , and that we are given pseudo-line bundles  $(E_{ij}, s_{ij})$  and a section  $u_{ijk}$  as above. Note that the  $E_{ij}$  are a collection of line bundles over intersections  $U_a \cap U_b$  where  $U_a \in \mathcal{U}_i$  and  $U_b \in \mathcal{U}_j$ . The gluing construction gives a Chatterjee-Hitchin gerbe  $(L, t)$  for the cover  $\mathcal{U}$  of  $M$ , where the  $E_{ij}$  enter the definition of transition line bundles between open sets in distinct  $\mathcal{U}_i, \mathcal{U}_j$ .

**REMARK 4.2.** Suppose  $X = M$ , and that all  $L_i, t_i, s_{ij}$  are trivial. Then the gerbe described in Proposition 4.1 is a Chatterjee-Hitchin gerbe for the cover  $\{V_i\}$ . The  $E_{ij}$  now play the role of transition line bundles, and  $u_{ijk}$  play the role of  $t$ .

Suppose now that, in addition to the assumptions of Proposition 4.1, we have gerbe connections  $(\nabla^{L_i}, B_i)$  and pseudo-line bundle connections  $\nabla^{E_{ij}} = (\nabla^{E_{ji}})^{-1}$ . Let  $\omega_{ij}$  denote the error 2-form for  $\nabla^{E_{ij}}$ .

**PROPOSITION 4.3.** *The connections  $\nabla^{L_{ij}} = \nabla^{L_j} \otimes \partial_1^* \nabla^{E_{ij}}$  on  $L_{ij}$ , together with the two forms  $B_i \in \Omega^2(X_i)$ , define a gerbe connection if all error 2-forms  $\omega_{ij}$  vanish, and if*

$$\nabla^{E_{ij}} \nabla^{E_{jk}} \nabla^{E_{ki}}(\pi^* u_{ijk}) = 0.$$

*Proof.* Let  $B$  be the 2-form on  $\coprod X_i$  given by  $B_i$  on  $X_i$ . We first verify that  $\frac{1}{2\pi i} \text{curv}(\nabla^{L_{ij}}) = (\delta B)_{ij}$ :

$$\begin{aligned} \frac{1}{2\pi i} \text{curv}(\nabla^{L_{ij}}) &= \frac{1}{2\pi i} \text{curv}(\nabla^{L_j}) + \frac{1}{2\pi i} \partial_1^* \text{curv}(\nabla^{E_{ij}}) \\ &= \delta B_j + \partial_1^*(B^j - B^i + \pi^* \omega_{ij}) \\ &= \partial_0^* B_j - \partial_1^* B_i = (\delta B)_{ij}. \end{aligned}$$

Next, we check that  $t_{ijk}$  is parallel for  $(\delta \nabla^L)_{ijk}$ :

$$\begin{aligned} (\delta \nabla^L)_{ijk} &= \partial_0^* \nabla^{L_{jk}} \partial_1^* (\nabla^{L_{ik}})^{-1} \partial_2^* \nabla^{L_{ij}} \\ &= \delta \nabla^{L_k} \otimes \partial_2^* (\nabla^{L_k} (\nabla^{L_j})^{-1} \delta \nabla^{E_{jk}}) \otimes \partial_2^* \partial_1^* (\nabla^{E_{ij}} \nabla^{E_{jk}} \nabla^{E_{ki}}). \end{aligned}$$

This annihilates (4.1) as required.

We now describe a slightly more complicated gluing construction, in which the  $X_i$  are not simply the restrictions of a surjective submersion  $X \rightarrow M$ . Instead, we assume that for each  $I$  we are given a surjective submersion  $\pi_I: X_I \rightarrow V_I$  are surjective submersions, and for each  $I \supset J$  a fiber preserving smooth map  $f_I^J: X_I \rightarrow X_J$ , with the compatibility condition  $f_J^K \circ f_I^J = f_I^K$  for  $I \supset J \supset K$ . Our gluing data will consist of the following:

- (i) Over each  $V_i$ , bundle gerbes  $(X_i, L_i, t_i)$  with connections  $(\nabla^{L_i}, B_i)$ .
- (ii) Over each  $V_{ij}$ , pseudo-line bundles  $E_{ij} = E_{ji}^{-1}, s_{ij} = s_{ji}^{-1}$  with connections  $\nabla^{E_{ij}} = (\nabla^{E_{ji}})^{-1}$  for the bundle gerbe  $(X_{ij}, L_{ij}, t_{ij})$ , given as the quotient of the pull-back of  $(X_j, L_j, t_j)$  by  $f_{ij}^j$  and the pull-back of  $(X_i, L_i, t_i)$  by  $f_{ij}^i$ .
- (iii) Over triple intersections, unitary sections  $u_{ijk}$  of the line bundle  $F_{ijk} \rightarrow V_{ijk}$  defined by tensoring the pull-backs of  $E_{ij}, E_{jk}, E_{ki}$  by the maps  $f_{ijk}^{ij}, f_{ijk}^{jk}, f_{ijk}^{ki}$ .

We require that the  $s_{ij}$  and  $u_{ijk}$  satisfy a cocycle condition similar to Proposition 4.1, that all error 2-forms  $\omega_{ij}$  are zero, and that the connections  $\nabla^{E_{ij}}$  satisfy a compatibility condition as in 4.3.

These data may be used to define a bundle gerbe over  $M$ , by reducing to the setting of Propositions 4.1, 4.3. As a first step we construct a more convenient cover.

LEMMA 4.4. *There are open subsets  $U_I$  of  $M$ , with  $\overline{U}_I \subset V_I$ , and  $\bigcup_I U_I = M$ , such that*

$$\overline{U}_I \cap \overline{U}_J = \emptyset \quad \text{unless} \quad J \subset I \text{ or } I \subset J.$$

*The collection of open subsets*

$$V'_i = M \setminus \bigcup_{J \ni i} \overline{U}_J$$

*is a shrinking of the open cover  $V_i$ , that is,  $\bigcup V'_i = M$  and  $\overline{V'_i} \subset V_i$ .*

The proof of this technical lemma is deferred to Appendix A. Now set  $X = \coprod_I X_I|_{U_I}$ . By definition of  $V'_i$ , the restriction  $X'_i = X|_{V'_i}$  is given by

$$X'_i = \coprod_{J \ni i} X_J|_{U_J \cap V'_i}.$$

More generally, letting  $V'_I = \bigcap_{i \in I} V'_i$  and  $X'_I = X|_{V'_I}$  we have

$$X'_I = \coprod_{J \supset I} X_J|_{U_J \cap V'_I}.$$

Let  $X'_I \rightarrow X_I|_{V'_I}$  be the fiber preserving map, given on  $X_J|_{U_J \cap V'_I}$  by the map  $f_J^I: X_J \rightarrow X_I$ . Using these maps, we can pull-back our gluing data: Let  $(X'_i, L'_i, t'_i)$  be the pull-back of the bundle gerbe  $(X_i, L_i, t_i)$  under the map  $X'_i \rightarrow X_i$ , equipped with the pull-back connection. On overlaps  $V'_{ij}$ , we let  $(E'_{ij}, s'_{ij})$  be the pseudo-line bundle with connections defined by pulling back  $(E_{ij}, s_{ij})$ . The gluing data obtained in this way satisfy the conditions from Propositions 4.1 and 4.3, and hence give rise to a bundle gerbe with connection over  $M$ .

REMARK 4.5. In our applications, the line bundles  $E_{ij}$  are in fact trivial, so one can simply take  $u_{ijk} = 1$  in terms of the trivialization. The  $s_{ij}$  are  $U(1)$ -valued functions in this case, and the compatibility condition reads  $s_{ij}s_{jk}s_{ki} = 1$  over  $X_{ijk}$ .

The gluing constructions generalize equivariant bundle gerbes in a straightforward way.