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REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \coprod U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G -action on X would amount to the cover being G -invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G , the equivariant cohomology $H_G^\bullet(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega_G^\bullet(M)$, consisting of G -equivariant polynomial maps $\alpha: \mathfrak{g} \rightarrow \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(d_G \alpha)(\xi) = d\alpha(\xi) - \iota(\xi_M)\alpha(\xi),$$

where $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G -equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G -closed equivariant curvature $\text{curv}_G(\nabla^L) \in \Omega_G^2(M)$.

A equivariant connection on a G -equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G) , where ∇^L is an invariant connection and $B_G \in \Omega_G^2(X)$ an equivariant 2-form, such that $\delta \nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i} \text{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ is defined by $\pi^* \eta_G = d_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s) , one defines the equivariant error 2-form ω_G by

$$\pi^* \omega_G = \frac{1}{2\pi i} \text{curv}_G(\nabla^E) - B_G.$$

Clearly, $d_G \omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G . Suppose $U(1) \rightarrow \widehat{K} \rightarrow K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. Given a principal K -bundle $\pi: P \rightarrow B$, one constructs a bundle gerbe (P, L, t) , sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K,$$

which we may view as a fiber bundle over B but also as a fiber bundle $E_n K \times_K P$ over $B_n K$. Let

$$(3.1) \quad f_\bullet: E_\bullet P \rightarrow B_\bullet K$$

be the bundle projection. Then $L = f_1^* \Gamma$, $t = f_2^* \tau$ defines a bundle gerbe (P, L, t) . A pseudo-line bundle for this bundle gerbe is equivalent to a lift of the structure group to \widehat{K} : Indeed if \widehat{P} is a principal \widehat{K} -bundle lifting P , consider the associated bundle $E = \widehat{P} \times_{U(1)} \mathbf{C}$. From the action map $\widehat{K} \times \widehat{P} \rightarrow \widehat{P}$ one obtains an isomorphism $\Gamma_k \otimes E_p \cong E_{k.p}$, or equivalently a section s of $\delta E^{-1} \otimes L$. One checks that $\delta s = t$, so that (E, s) is a pseudo-line bundle. Conversely, the bundle \widehat{P} is recovered as the unit circle bundle in E , and s defines an action of \widehat{K} lifting the action of K . See Gomi [14] for a detailed construction of bundle gerbe connections on (P, L, t) .

REMARK 3.1. To obtain a Chatterjee-Hitchin gerbe from this bundle gerbe, we must choose a cover \mathcal{U} of M such that P is trivial over each $U_a \in \mathcal{U}$. Any choice of trivialization gives a simplicial map $\mathcal{U}_\bullet M \rightarrow E_\bullet P$, and we pull back the bundle gerbe under this map. More directly, the local trivializations give rise to a 'classifying map' $\chi_\bullet: \mathcal{U}_\bullet M \rightarrow B_\bullet K$ (see [23]), and the Chatterjee-Hitchin gerbe is defined as the pull-back of (Γ, τ) under this map.

Suppose the group K is compact and connected. After pulling back to the universal cover \widetilde{K} , every central extension $U(1) \rightarrow \widehat{K} \rightarrow K$ becomes trivial. It follows that every central extension of K by $U(1)$ is of the form

$$\widehat{K} = \widetilde{K} \times_{\pi_1(K)} U(1),$$

where $\pi_1(K) \subset \widetilde{K}$ acts on $U(1)$ via some homomorphism $\varrho \in \text{Hom}(\pi_1(K), U(1))$. The choice of ϱ for a given extension is equivalent to the choice of a flat \widehat{K} -invariant connection on the principal $U(1)$ -bundle $\widehat{K} \rightarrow K$. The central extension is isomorphic to the *trivial* extension if and only if ϱ extends to a homomorphism $\widetilde{\varrho}: \widetilde{K} \rightarrow U(1)$, and the choice of any such $\widetilde{\varrho}$ is equivalent to a choice of trivialization. Using the natural map from $(\mathfrak{k}^*)^K = \text{Hom}(\widetilde{K}, \mathbf{R})$ onto $\text{Hom}(\widetilde{K}, U(1))$ this gives an exact sequence of Abelian groups

$$(3.2) \quad (\mathfrak{k}^*)^K \rightarrow \text{Hom}(\pi_1(K), U(1)) \rightarrow \{\text{central extensions of } K \text{ by } U(1)\} \rightarrow 1.$$

Suppose K is semi-simple (so that $(\mathfrak{k}^*)^K = 0$), and T is a maximal torus in K . Let $\widetilde{T} \subset \widetilde{K}$ be the maximal torus given as the pre-image of T . Let $\Lambda_K, \widetilde{\Lambda}_K \subset \mathfrak{t}$ be the integral lattices of T, \widetilde{T} . The lattice $\widetilde{\Lambda}_K$ is equal to the

co-root lattice of K , and $\pi_1(K) = \Lambda_K / \tilde{\Lambda}_K$ (cf. [6, Theorem V.7.1]). Therefore, if K is semi-simple,

$$\{\text{central extensions of } K \text{ by } U(1)\} = \text{Hom}(\pi_1(K), U(1)) = \tilde{\Lambda}_K^* / \Lambda_K^*,$$

the quotient of the dual of the co-root lattice by the weight lattice.

PROPOSITION 3.2. *Suppose K is a compact, connected Lie group and $\pi: P \rightarrow M$ a principal K -bundle.*

(a) *Any $\varrho \in \text{Hom}(\pi_1(K), U(1))$ defines a bundle gerbe (P, L, t) over M , together with a gerbe connection (∇^L, B) where $B = 0$. In particular this gerbe is flat.*

(b) *If ϱ is the image of $\mu \in (\mathfrak{k}^*)^K$, there is a distinguished pseudo-line bundle $\mathcal{L} = (E, s)$ for this gerbe, with E a trivial line bundle. Any principal connection $\theta \in \Omega^1(P, \mathfrak{k})$ defines a connection on \mathcal{L} , with error 2-form $\omega \in \Omega^2(M)$ given by $\pi^*\omega = \langle \mu, F^\theta \rangle \in \Omega^2(M)$, where F^θ is the curvature.*

Proof. Let $U(1) \rightarrow \hat{K} \rightarrow K$ be the central extension defined by ϱ , and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. As remarked above, ϱ defines a flat connection on $\hat{K} \rightarrow K$, hence also a flat connection ∇^Γ on the line bundle $\Gamma \rightarrow B_1 K$. Then $(\nabla^\Gamma, 0)$ is a connection on the simplicial gerbe (Γ, τ) . Pulling back under the map f_* (cf. (3.1)) we obtain a connection $(\nabla^L, 0)$ on the bundle gerbe (P, L, t) .

If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, the corresponding trivialization of \hat{K} defines a unitary section σ of Γ , with $\delta\sigma = \tau$ and $\frac{1}{2\pi i} \nabla^\Gamma \sigma = \langle \mu, \theta^L \rangle \sigma$, where θ^L is the left-invariant Maurer-Cartan form on K . Thus $\mathcal{L} = (E, s)$, with E the trivial line bundle and $s = f_1^* \sigma$, is a pseudo-line bundle for \mathcal{G} . Given a principal connection θ , let ∇^E be the connection on the trivial bundle E , having connection 1-form $\langle \mu, \theta \rangle \in \Omega^1(P)$. Since $\frac{1}{2\pi i} \nabla^L s = f_1^* \langle \mu, \theta^L \rangle s$, it follows that

$$(3.3) \quad \frac{1}{2\pi i} ((\delta \nabla^E)^{-1} \nabla^L) s = \langle \mu, f_1^* \theta^L - \delta \theta \rangle.$$

One finds $\partial_1^* \theta = \text{Ad}_{f_1^{-1}}(\partial_0^* \theta - f_1^* \theta^L)$. Since μ is K -invariant, this shows that the right hand side of (3.3) vanishes. Thus ∇^E is a pseudo-line bundle connection. The error 2-form ω is given by

$$\pi^* \omega = d \langle \mu, \theta \rangle = \langle \mu, d\theta \rangle = \langle \mu, F^\theta \rangle.$$

All of these constructions can be made equivariant in a rather obvious way: Thus if G is another Lie group and P is a G -invariant principal K -bundle, any $\varrho \in \text{Hom}(\pi_1(K), \text{U}(1))$ defines a G -equivariant bundle gerbe (P, L, t) (with flat connection) over M . If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, there is a G -equivariant pseudo-line bundle for this gerbe. Furthermore any choice of G -equivariant principal connection on P defines a G -equivariant pseudo-line bundle connection, with equivariant error 2-form $\pi^*\omega_G = \langle \mu, F_G^\theta \rangle$ where $F_G^\theta \in \Omega_G^2(P, \mathfrak{k})$ is the equivariant curvature.

4. GLUING DATA

In this Section we describe a procedure for gluing a collection of bundle gerbes (X_i, L_i, t_i) on open subsets $V_i \subset M$, with pseudo-line bundles of their quotients on overlaps²). We begin with the somewhat simpler case that the surjective submersions $X_i \rightarrow V_i$ are obtained by restricting a surjective submersion $X \rightarrow M$, and later reduce the general case to this special case.

Thus, let $\pi: X \rightarrow M$ be a surjective submersion and let V_i , $i = 0, \dots, d$ an open cover of M . Let $X_i = X|_{V_i}$, and more generally $X_I = X|_{V_I}$ where V_I is the intersection of all V_i with $i \in I$.

Suppose we are given bundle gerbes (X_i, L_i, t_i) over V_i and pseudo-line bundles (E_{ij}, s_{ij}) for the quotients $(X_{ij}, L_j L_i^{-1}, t_j t_i^{-1})$ over $V_i \cap V_j$, where $E_{ij} = E_{ji}^{-1}$ and $s_{ij} = s_{ji}^{-1}$. Note that $E_{ij} E_{jk} E_{ki}$ is a pseudo-line bundle for the trivial gerbe, hence is a pull-back $\pi^* F_{ijk}$ of a line bundle $F_{ijk} \rightarrow M$, and we will also require a unitary section u_{ijk} of that line bundle. Under suitable conditions the data (E_{ij}, s_{ij}) and u_{ijk} can be used to 'glue' the gerbes (X_i, L_i, t_i) . The glued gerbe will be defined over the disjoint union $\coprod_{i=1}^d X_i$. We have

$$\begin{aligned} \left(\coprod_{i=1}^d X_i \right)^{[2]} &= \coprod_{ij} X_i \times_M X_j \\ \left(\coprod_{i=1}^d X_i \right)^{[3]} &= \coprod_{ijk} X_i \times_M X_j \times_M X_k \\ &\dots \end{aligned}$$

Hence, the glued gerbe will be of the form $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ where L_{ij} are line bundles over $X_i \times_M X_j$ and t_{ijk} unitary sections of a line bundle $(\delta L)_{ijk}$

²) See Stevenson [29] for similar gluing constructions.