

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE BASIC GERBE OVER A COMPACT SIMPLE LIE GROUP
Autor: Meinrenken, Eckhard
Kapitel: 3. Gerbes from principal bundles
DOI: <https://doi.org/10.5169/seals-66691>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \bigsqcup U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G -action on X would amount to the cover being G -invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G , the equivariant cohomology $H_G^\bullet(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega_G^\bullet(M)$, consisting of G -equivariant polynomial maps $\alpha: \mathfrak{g} \rightarrow \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(d_G \alpha)(\xi) = d \alpha(\xi) - \iota(\xi_M) \alpha(\xi),$$

where $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G -equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G -closed equivariant curvature $\text{curv}_G(\nabla^L) \in \Omega_G^2(M)$.

A equivariant connection on a G -equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G) , where ∇^L is an invariant connection and $B_G \in \Omega_G^2(X)$ an equivariant 2-form, such that $\delta \nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i} \text{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ is defined by $\pi^* \eta_G = d_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s) , one defines the equivariant error 2-form ω_G by

$$\pi^* \omega_G = \frac{1}{2\pi i} \text{curv}_G(\nabla^E) - B_G.$$

Clearly, $d_G \omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G . Suppose $\text{U}(1) \rightarrow \widehat{K} \rightarrow K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over BK . Given a principal K -bundle $\pi: P \rightarrow B$, one constructs a bundle gerbe (P, L, t) , sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K,$$

which we may view as a fiber bundle over B but also as a fiber bundle $E_n K \times_K P$ over $B_n K$. Let

$$(3.1) \quad f_\bullet: E_\bullet P \rightarrow B_\bullet K$$

be the bundle projection. Then $L = f_1^* \Gamma$, $t = f_2^* \tau$ defines a bundle gerbe (P, L, t) . A pseudo-line bundle for this bundle gerbe is equivalent to a lift of the structure group to \widehat{K} : Indeed if \widehat{P} is a principal \widehat{K} -bundle lifting P , consider the associated bundle $E = \widehat{P} \times_{U(1)} \mathbf{C}$. From the action map $\widehat{K} \times \widehat{P} \rightarrow \widehat{P}$ one obtains an isomorphism $\Gamma_k \otimes E_p \cong E_{k,p}$, or equivalently a section s of $\delta E^{-1} \otimes L$. One checks that $\delta s = t$, so that (E, s) is a pseudo-line bundle. Conversely, the bundle \widehat{P} is recovered as the unit circle bundle in E , and s defines an action of \widehat{K} lifting the action of K . See Gomi [14] for a detailed construction of bundle gerbe connections on (P, L, t) .

REMARK 3.1. To obtain a Chatterjee-Hitchin gerbe from this bundle gerbe, we must choose a cover \mathcal{U} of M such that P is trivial over each $U_a \in \mathcal{U}$. Any choice of trivialization gives a simplicial map $\mathcal{U}M \rightarrow E_\bullet P$, and we pull back the bundle gerbe under this map. More directly, the local trivializations give rise to a 'classifying map' $\chi_\bullet: \mathcal{U}M \rightarrow B_\bullet K$ (see [23]), and the Chatterjee-Hitchin gerbe is defined as the pull-back of (Γ, τ) under this map.

Suppose the group K is compact and connected. After pulling back to the universal cover \widetilde{K} , every central extension $U(1) \rightarrow \widehat{K} \rightarrow K$ becomes trivial. It follows that every central extension of K by $U(1)$ is of the form

$$\widehat{K} = \widetilde{K} \times_{\pi_1(K)} U(1),$$

where $\pi_1(K) \subset \widetilde{K}$ acts on $U(1)$ via some homomorphism $\varrho \in \text{Hom}(\pi_1(K), U(1))$. The choice of ϱ for a given extension is equivalent to the choice of a flat \widehat{K} -invariant connection on the principal $U(1)$ -bundle $\widehat{K} \rightarrow K$. The central extension is isomorphic to the *trivial* extension if and only if ϱ extends to a homomorphism $\widetilde{\varrho}: \widetilde{K} \rightarrow U(1)$, and the choice of any such $\widetilde{\varrho}$ is equivalent to a choice of trivialization. Using the natural map from $(\mathfrak{k}^*)^K = \text{Hom}(\widetilde{K}, \mathbf{R})$ onto $\text{Hom}(\widetilde{K}, U(1))$ this gives an exact sequence of Abelian groups

$$(3.2) \quad (\mathfrak{k}^*)^K \rightarrow \text{Hom}(\pi_1(K), U(1)) \rightarrow \{\text{central extensions of } K \text{ by } U(1)\} \rightarrow 1.$$

Suppose K is semi-simple (so that $(\mathfrak{k}^*)^K = 0$), and T is a maximal torus in K . Let $\widetilde{T} \subset \widetilde{K}$ be the maximal torus given as the pre-image of T . Let $\Lambda_K, \widetilde{\Lambda}_K \subset \mathfrak{t}$ be the integral lattices of T, \widetilde{T} . The lattice $\widetilde{\Lambda}_K$ is equal to the

co-root lattice of K , and $\pi_1(K) = \Lambda_K/\tilde{\Lambda}_K$ (cf. [6, Theorem V.7.1]). Therefore, if K is semi-simple,

$$\{\text{central extensions of } K \text{ by } \text{U}(1)\} = \text{Hom}(\pi_1(K), \text{U}(1)) = \tilde{\Lambda}_K^*/\Lambda_K^*,$$

the quotient of the dual of the co-root lattice by the weight lattice.

PROPOSITION 3.2. *Suppose K is a compact, connected Lie group and $\pi: P \rightarrow M$ a principal K -bundle.*

(a) *Any $\varrho \in \text{Hom}(\pi_1(K), \text{U}(1))$ defines a bundle gerbe (P, L, t) over M , together with a gerbe connection (∇^L, B) where $B = 0$. In particular this gerbe is flat.*

(b) *If ϱ is the image of $\mu \in (\mathfrak{k}^*)^K$, there is a distinguished pseudo-line bundle $\mathcal{L} = (E, s)$ for this gerbe, with E a trivial line bundle. Any principal connection $\theta \in \Omega^1(P, \mathfrak{k})$ defines a connection on \mathcal{L} , with error 2-form $\omega \in \Omega^2(M)$ given by $\pi^*\omega = \langle \mu, F^\theta \rangle \in \Omega^2(M)$, where F^θ is the curvature.*

Proof. Let $\text{U}(1) \rightarrow \widehat{K} \rightarrow K$ be the central extension defined by ϱ , and (Γ, τ) the corresponding simplicial gerbe over BK . As remarked above, ϱ defines a flat connection on $\widehat{K} \rightarrow K$, hence also a flat connection ∇^Γ on the line bundle $\Gamma \rightarrow B_1 K$. Then $(\nabla^\Gamma, 0)$ is a connection on the simplicial gerbe (Γ, τ) . Pulling back under the map f_* (cf. (3.1)) we obtain a connection $(\nabla^L, 0)$ on the bundle gerbe (P, L, t) .

If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, the corresponding trivialization of \widehat{K} defines a unitary section σ of Γ , with $\delta\sigma = \tau$ and $\frac{1}{2\pi i} \nabla^\Gamma \sigma = \langle \mu, \theta^L \rangle \sigma$, where θ^L is the left-invariant Maurer-Cartan form on K . Thus $\mathcal{L} = (E, s)$, with E the trivial line bundle and $s = f_1^* \sigma$, is a pseudo-line bundle for \mathcal{G} . Given a principal connection θ , let ∇^E be the connection on the trivial bundle E , having connection 1-form $\langle \mu, \theta \rangle \in \Omega^1(P)$. Since $\frac{1}{2\pi i} \nabla^L s = f_1^* \langle \mu, \theta^L \rangle s$, it follows that

$$(3.3) \quad \frac{1}{2\pi i} ((\delta \nabla^E)^{-1} \nabla^L) s = \langle \mu, f_1^* \theta^L - \delta \theta \rangle.$$

One finds $\partial_1^* \theta = \text{Ad}_{f_1^{-1}}(\partial_0^* \theta - f_1^* \theta^L)$. Since μ is K -invariant, this shows that the right hand side of (3.3) vanishes. Thus ∇^E is a pseudo-line bundle connection. The error 2-form ω is given by

$$\pi^* \omega = d \langle \mu, \theta \rangle = \langle \mu, d \theta \rangle = \langle \mu, F^\theta \rangle.$$

All of these constructions can be made equivariant in a rather obvious way: Thus if G is another Lie group and P is a G -invariant principal K -bundle, any $\varrho \in \text{Hom}(\pi_1(K), \text{U}(1))$ defines a G -equivariant bundle gerbe (P, L, t) (with flat connection) over M . If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, there is a G -equivariant pseudo-line bundle for this gerbe. Furthermore any choice of G -equivariant principal connection on P defines a G -equivariant pseudo-line bundle connection, with equivariant error 2-form $\pi^* \omega_G = \langle \mu, F_G^\theta \rangle$ where $F_G^\theta \in \Omega_G^2(P, \mathfrak{k})$ is the equivariant curvature.

4. GLUING DATA

In this Section we describe a procedure for gluing a collection of bundle gerbes (X_i, L_i, t_i) on open subsets $V_i \subset M$, with pseudo-line bundles of their quotients on overlaps²⁾. We begin with the somewhat simpler case that the surjective submersions $X_i \rightarrow V_i$ are obtained by restricting a surjective submersion $X \rightarrow M$, and later reduce the general case to this special case.

Thus, let $\pi: X \rightarrow M$ be a surjective submersion and let V_i , $i = 0, \dots, d$ an open cover of M . Let $X_i = X|_{V_i}$, and more generally $X_I = X|_{V_I}$ where V_I is the intersection of all V_i with $i \in I$.

Suppose we are given bundle gerbes (X_i, L_i, t_i) over V_i and pseudo-line bundles (E_{ij}, s_{ij}) for the quotients $(X_{ij}, L_j L_i^{-1}, t_j t_i^{-1})$ over $V_i \cap V_j$, where $E_{ij} = E_{ji}^{-1}$ and $s_{ij} = s_{ji}^{-1}$. Note that $E_{ij} E_{jk} E_{ki}$ is a pseudo-line bundle for the trivial gerbe, hence is a pull-back $\pi^* F_{ijk}$ of a line bundle $F_{ijk} \rightarrow M$, and we will also require a unitary section u_{ijk} of that line bundle. Under suitable conditions the data (E_{ij}, s_{ij}) and u_{ijk} can be used to 'glue' the gerbes (X_i, L_i, t_i) . The glued gerbe will be defined over the disjoint union $\coprod_{i=1}^d X_i$. We have

$$\begin{aligned} \left(\coprod_{i=1}^d X_i \right)^{[2]} &= \coprod_{ij} X_i \times_M X_j \\ \left(\coprod_{i=1}^d X_i \right)^{[3]} &= \coprod_{ijk} X_i \times_M X_j \times_M X_k \\ &\dots \end{aligned}$$

Hence, the glued gerbe will be of the form $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ where L_{ij} are line bundles over $X_i \times_M X_j$ and t_{ijk} unitary sections of a line bundle $(\delta L)_{ijk}$

²⁾ See Stevenson [29] for similar gluing constructions.