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Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the  $\delta$ -cohomology of the double complex  $\Omega^k(M_n)$  vanishes in bidegrees  $(1, 2)$  and  $(2, 1)$ . In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion  $\pi: X \rightarrow M$  the sequence

$$(2.1) \quad 0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \dots$$

is exact, so the  $\delta$ -cohomology vanishes in *all* degrees.

Thus, every bundle gerbe  $\mathcal{G} = (X, L, t)$  over a manifold  $M$  (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the *3-curvature*  $\eta \in \Omega^3(M)$  of the bundle gerbe connection by  $\pi^*\eta = dB \in \ker \delta$ . It can be shown that its cohomology class is the image of the Dixmier-Douady class  $[\mathcal{G}]$  under the map  $H^3(M, \mathbf{Z}) \rightarrow H^3(M, \mathbf{R})$ . Similarly, if  $\mathcal{G}$  admits a pseudo-line bundle  $\mathcal{L} = (E, s)$ , one can always choose a pseudo-line bundle connection  $\nabla^E$ . The difference  $\frac{1}{2\pi i} \text{curv}(\nabla^E) - B$  is  $\delta$ -closed and one defines the *error 2-form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i} \text{curv}(\nabla^E) - B.$$

It is clear from the definition that  $d\omega + \eta = 0$ .

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

## 2.4 EQUIVARIANT BUNDLE GERBES

Suppose  $G$  is a Lie group acting on  $X$  and on  $M$ , and that  $\pi: X \rightarrow M$  is a  $G$ -equivariant surjective submersion. Then  $G$  acts on all fiber products  $X^{[p]}$ . We will say that a bundle gerbe  $\mathcal{G} = (X, L, t)$  is *G-equivariant*, if  $L$  is a  $G$ -equivariant line bundle and  $t$  is a  $G$ -invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction<sup>1)</sup>  $X_G = EG \times_G X \rightarrow M_G = EG \times_G M$ , hence has an *equivariant* Dixmier-Douady class in  $H^3(M_G, \mathbf{Z}) = H_G^3(M, \mathbf{Z})$ . Similarly, we say that a pseudo-line bundle  $(E, s)$  for  $(X, L, t)$  is equivariant, provided  $E$  carries a  $G$ -action and  $s$  is an invariant section.

<sup>1)</sup> We have not discussed bundle gerbes over infinite-dimensional spaces such as  $M_G$ . Recall however [4] that the classifying bundle  $EG \rightarrow BG$  may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if  $X = \coprod U_a$ , for an open cover  $\mathcal{U} = \{U_a, a \in A\}$ , a  $G$ -action on  $X$  would amount to the cover being  $G$ -invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group  $G$ , the equivariant cohomology  $H_G^\bullet(M, \mathbf{R})$  may be computed from Cartan's complex of equivariant differential forms  $\Omega_G^\bullet(M)$ , consisting of  $G$ -equivariant polynomial maps  $\alpha: \mathfrak{g} \rightarrow \Omega(M)$ . The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(d_G \alpha)(\xi) = d\alpha(\xi) - \iota(\xi_M)\alpha(\xi),$$

where  $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$  is the generating vector field corresponding to  $\xi \in \mathfrak{g}$ . Given a  $G$ -equivariant connection  $\nabla^L$  on an equivariant line bundle, one defines [3, Chapter 7] a  $d_G$ -closed equivariant curvature  $\text{curv}_G(\nabla^L) \in \Omega_G^2(M)$ .

A equivariant connection on a  $G$ -equivariant bundle gerbe  $(X, L, t)$  over  $M$  is a pair  $(\nabla^L, B_G)$ , where  $\nabla^L$  is an invariant connection and  $B_G \in \Omega_G^2(X)$  an equivariant 2-form, such that  $\delta \nabla^L t = 0$  and  $\delta B_G = \frac{1}{2\pi i} \text{curv}_G(\nabla^L)$ . Its equivariant 3-curvature  $\eta_G \in \Omega_G^3(M)$  is defined by  $\pi^* \eta_G = d_G B_G$ . Given an *invariant* pseudo-line bundle connection  $\nabla^E$  on a equivariant pseudo-line bundle  $(E, s)$ , one defines the equivariant error 2-form  $\omega_G$  by

$$\pi^* \omega_G = \frac{1}{2\pi i} \text{curv}_G(\nabla^E) - B_G.$$

Clearly,  $d_G \omega_G + \eta_G = 0$ .

### 3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over  $G$ . Suppose  $U(1) \rightarrow \widehat{K} \rightarrow K$  is a central extension, and  $(\Gamma, \tau)$  the corresponding simplicial gerbe over  $B_\bullet K$ . Given a principal  $K$ -bundle  $\pi: P \rightarrow B$ , one constructs a bundle gerbe  $(P, L, t)$ , sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K,$$