Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE BASIC GERBE OVER A COMPACT SIMPLE LIE GROUP
Autor: Meinrenken, Eckhard
Kapitel: 2.4 Equivariant bundle gerbes
DOI: https://doi.org/10.5169/seals-66691

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 15.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the δ-cohomology of the double complex $\Omega^k(M_n)$ vanishes in bidegrees $(1,2)$ and $(2,1)$. In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion $\pi\colon X \to M$ the sequence

$$(2.1) \qquad 0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \cdots$$

is exact, so the δ-cohomology vanishes in *all* degrees.

Thus, every bundle gerbe $\mathcal{G} = (X, L, t)$ over a manifold M (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the 3-*curvature* $\eta \in \Omega^3(M)$ of the bundle gerbe connection by $\pi^*\eta = \mathrm{d}B \in \ker\delta$. It can be shown that its cohomology class is the image of the Dixmier-Douady class $[\mathcal{G}]$ under the map $H^3(M, \mathbf{Z}) \to H^3(M, \mathbf{R})$. Similarly, if $\mathcal{G}$ admits a pseudo-line bundle $\mathcal{L} = (E, s)$, one can always choose a pseudo-line bundle connection ∇^E. The difference $\frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B$ is δ-closed and one defines the *error* 2-*form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B\,.$$

It is clear from the definition that $\mathrm{d}\omega + \eta = 0$.

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

2.4 EQUIVARIANT BUNDLE GERBES

Suppose G is a Lie group acting on X and on M, and that $\pi\colon X \to M$ is a G-equivariant surjective submersion. Then G acts on all fiber products $X^{[p]}$. We will say that a bundle gerbe $\mathcal{G} = (X, L, t)$ is G-*equivariant*, if L is a G-equivariant line bundle and t is a G-invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction[1]) $X_G = EG \times_G X \to M_G = EG \times_G M$, hence has an *equivariant* Dixmier-Douady class in $H^3(M_G, \mathbf{Z}) = H^3_G(M, \mathbf{Z})$. Similarly, we say that a pseudo-line bundle (E, s) for (X, L, t) is equivariant, provided E carries a G-action and s is an invariant section.

[1]) We have not discussed bundle gerbes over infinite-dimensional spaces such as M_G. Recall however [4] that the classifying bundle $EG \to BG$ may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \coprod U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G-action on X would amount to the cover being G-invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G, the equivariant cohomology $H^\bullet_G(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega^\bullet_G(M)$, consisting of G-equivariant polynomial maps $\alpha\colon \mathfrak{g} \to \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(\mathrm{d}_G\,\alpha)(\xi) = \mathrm{d}\,\alpha(\xi) - \iota(\xi_M)\alpha(\xi)\,,$$

where $\xi_M = \frac{d}{dt}|_{t=0}\exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G-equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G-closed equivariant curvature $\mathrm{curv}_G(\nabla^L) \in \Omega^2_G(M)$.

A equivariant connection on a G-equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G), where ∇^L is an invariant connection and $B_G \in \Omega^2_G(X)$ an equivariant 2-form, such that $\delta\nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega^3_G(M)$ is defined by $\pi^*\eta_G = \mathrm{d}_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s), one defines the equivariant error 2-form ω_G by

$$\pi^*\omega_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^E) - B_G\,.$$

Clearly, $\mathrm{d}_G\,\omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G. Suppose $\mathrm{U}(1) \to \widehat{K} \to K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. Given a principal K-bundle $\pi\colon P \to B$, one constructs a bundle gerbe (P, L, t), sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K\,,$$