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over $X^{[3]}$, satisfying a compatibility condition $\delta t=1$ over $X^{[4]}$ (which makes sense since δt is a section of the canonically trivial bundle $\delta \delta L$). Given local sections $\sigma_a \colon U_a \to X$, one can pull these data back under the maps $(\sigma_a,\sigma_b)\colon U_a\cap U_b\to X^{[2]}$ and $(\sigma_a,\sigma_b,\sigma_c)\colon U_a\cap U_b\cap U_c\to X^{[3]}$ to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of (X,L,t) is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with X the disjoint union of the sets U_a in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called 'locally split') is used that every point $x \in M$ admits an open neighborhood U and a map $\sigma \colon U \to X$ such that $\pi \circ \sigma = \mathrm{id}$. However, this condition seems insufficient in the smooth category, as the fiber product $X \times_M X$ need not be a manifold unless π is a submersion.

2.3 SIMPLICIAL GERBES

Murray's construction fits naturally into a wider context of *simplicial* gerbes. We refer to Mostow-Perchik's notes of lectures by R. Bott [23] and to Dupont's paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold* M_{\bullet} is a sequence of manifolds $(M_n)_{n=0}^{\infty}$, together with *face maps* $\partial_i : M_n \to M_{n-1}$ for $i = 0, \ldots, n$ satisfying relations $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for i < j. (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of M_{\bullet} is the topological space $||M|| = \coprod_{n=1}^{\infty} \Delta^n \times M_n / \sim$, where Δ^n is the *n*-simplex and the relation is $(t, \partial_i(x)) \sim (\partial^i(t), x)$, for $\partial^i : \Delta^{n-1} \to \Delta^n$ the inclusion as the *i*th face. A (smooth) simplicial map between simplicial manifolds $M_{\bullet}, M'_{\bullet}$ is a collection of smooth maps $f_n : M_n \to M'_n$ intertwining the face maps; such a map induces a map between the geometric realizations.

EXAMPLES 2.2.

(a) If S is any manifold, one can define a simplicial manifold $E_{\bullet}S$ where E_nS is the n+1-fold cartesian product of S, and ∂_j omits the jth factor. It is known [23] that the geometric realization ||ES|| of this simplicial manifold is contractible. More generally, if $X \to M$ is a fiber bundle with fiber S,

one can define a simplicial manifold $E_nX := X^{[n+1]}$, with face maps as in Section 2.2. The geometric realization ||EX|| becomes a fiber bundle over M with contractible fiber ||ES||.

(b) [22, 27] For any Lie group G there is a simplicial manifold $B_nG = G^n$. The face maps ∂_i for 0 < i < n are

$$\partial_i(g_1,\ldots,g_n)=(g_1,\ldots,g_ig_{i+1},\ldots,g_n),$$

while ∂_0 omits the first component and ∂_n the last component. The map $\pi_n: E_nG \to B_nG$ given by $\pi_n(k_0, \ldots, k_n) = (k_0k_1^{-1}, \ldots, k_{n-1}k_n^{-1})$ is simplicial, and the induced map on geometric realizations is a model for the classifying bundle $EG \to BG$.

(c) [27, 23] If $\mathcal{U} = \{U_a, a \in A\}$ is an open cover of M, one defines a simplicial manifold

$$\mathcal{U}_n M := \coprod_{(a_0,\ldots,a_n)\in A_n} U_{a_0\ldots a_n}$$

where A_n is the set of all sequences (a_0, \ldots, a_n) such that $U_{a_0 \ldots a_n} := U_{a_0} \cap \ldots \cap U_{a_n}$ is non-empty. The face maps are induced by the inclusions,

$$\partial_i \colon U_{a_0 \dots a_n} \hookrightarrow U_{a_0 \dots \widehat{a_i} \dots a_n}$$
.

One may view this as a special case of (a), with $X = \coprod_{a \in A} U_a$. It is known [23, Theorem 7.3] that $||\mathcal{U}M||$ is homotopy equivalent to M.

(d) [2] The definitions of E_nG and B_nG extend to Lie groupoids G over a base S. If $s,t\colon G\to S$ are the source and target maps, one defines E_nG as the n+1-fold fiber product of G with respect to the target map t. The space B_nG for $n\geq 1$ is the set of all $(g_1,\ldots,g_n)\in G^n$ with $s(g_j)=t(g_{j-1})$, while $B_0G=S$. The definition of the face maps $\partial_j\colon B_nG\to B_{n-1}G$ is as before for n>1, while for n=1, $\partial_0=t$ and $\partial_1=s$. We have a simplicial map $E_nG\to B_nG$ defined just as in the group case.

The bi-graded space of differential forms $\Omega^{\bullet}(M_{\bullet})$ carries two commuting differentials d, δ , where d is the de Rham differential and $\delta \colon \Omega^k(M_n) \to \Omega^k(M_{n+1})$ is an alternating sum, $\delta \alpha = \sum_{i=0}^{n+1} (-1)^i \partial_i^* \alpha$. It is known [23, Theorem 4.2, Theorem 4.5] that the total cohomology of this double complex is the (singular) cohomology of the geometric realization, with coefficients in \mathbf{R} .

We will use the δ notation in many similar situations: For instance, given a Hermitian line bundle $L \to M_n$, we define a Hermitian line bundle $\delta L \to M_{n+1}$ as a tensor product,

$$\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \cdots \otimes \partial_{n+1}^* L^{\pm}.$$

The line bundle $\delta(\delta L) \to M_{n+1}$ is canonically trivial, due to the relations between face maps. If σ is a unitary section (i.e. a trivialization) of L, one uses a similar formula to define a unitary section $\delta \sigma$ of δL . Then $\delta(\delta \sigma) = 1$ (the identity section of the trivial line bundle $\delta(\delta L)$). For any unitary connection ∇ of L, one defines a unitary connection $\delta \nabla$ of δL in the obvious way.

CONVENTION. For the rest of this paper, we take all line bundles L to be *Hermitian* line bundles, and all connections ∇ on L to be *unitary* connections.

Let M_{ullet} be a simplicial manifold. One might define a simplicial line bundle as a collection of line bundles $L_n \to M_n$ such that the face maps $\partial_i \colon M_n \to M_{n-1}$ lift to line bundle homomorphisms $\hat{\partial}_i \colon L_n \to L_{n-1}$, satisfying the face map relations. Thus L_{ullet} is itself a simplicial manifold, and its geometric realization ||L|| is a line bundle over ||M||. Equivalently, the lifts $\hat{\partial}_i$ may be viewed as isomorphisms, $\partial_i^* L_{n-1} \to L_n$. In particular, we may identify L_n with the pull-back of $L := L_0$ under the nth-fold iterate $\partial_0 \circ \cdots \circ \partial_0$.

The isomorphisms $\partial_1^*L \cong \partial_0^*L = L_1$ determine a unitary section t of $\delta L \to M_1$, and the compatibility of isomorphisms

$$(\partial_0 \partial_2)^* L \cong (\partial_0 \partial_1)^* L \cong (\partial_0 \partial_0)^* L = L_2$$

amount to the condition $\delta t = 1$. (Compatibility of the isomorphisms for L_n with $n \geq 3$ is then automatic.) That is, a simplicial line bundle over M_{\bullet} is given by a line bundle $L \to M_0$, together with a unitary section t of $\delta L \to M_1$, such that $\delta t = 1$ over M_2 . A unitary section s of L with $\delta s = t$ induces a unitary section of $||L|| \to ||M||$.

Taking L to be trivial, we see in particular that any U(1)-valued function t on M_1 , with $\delta t=1$, defines a line bundle over the geometric realization. A trivialization of that line bundle is given by a U(1)-valued function on M_0 satisfying $\delta s=t$. Replacing U(1)-valued functions with line bundles, this motivates the following definition.

DEFINITION 2.3. A simplicial gerbe over M_{\bullet} is a pair (L,t), consisting of a line bundle $L \to M_1$, together with a section t of $\delta L \to M_2$ satisfying $\delta t = 1$. A pseudo-line bundle for (L,t) is a pair (E,s), consisting of a line bundle $E \to M_0$ and a section s of $\delta E^{-1} \otimes L$ such that $\delta s = t$.

REMARK 2.4.

- (a) We are using the notion of a simplicial gerbe only as a 'working definition'. It is clear from the discussion above that a more general notion would involve a gerbe over M_0 .
- (b) In [9], what we call simplicial gerbe is called a simplicial line bundle. The name pseudo-line bundle is adopted from [9], where it is used in a similar context.

A simplicial gerbe over $\mathcal{U}_{\bullet}M$ (for a cover \mathcal{U} of M) is a Chatterjee-Hitchin gerbe, while a simplicial gerbe over $E_{\bullet}X = X^{[\bullet+1]}$ (for a surjective submersion $X \to M$) is a bundle gerbe. It is shown in [24] that the characteristic class of a bundle gerbe (X, L, t) vanishes if and only if it admits a pseudo-line bundle.

EXAMPLE 2.5 (Central extensions). (See [9, p.615].) Let K be a Lie group. A simplicial line bundle over $B_{\bullet}K$ is the same thing as a group homomorphism $K \to U(1)$: The line bundle $L \to B_0K$ is trivial since B_0K is just a point, hence the unitary section t of δL becomes a U(1)-valued function. The condition $\delta t = 1$ means that this function is a group homomorphism.

Similarly, a simplicial gerbe (Γ, τ) over $B_{\bullet}K$ is the same thing as a central extension

$$U(1) \rightarrow \widehat{K} \rightarrow K$$
.

Indeed, given the line bundle $\Gamma \to K$ let \widehat{K} be the unit circle bundle inside Γ . The fiber of $\delta\Gamma \to K^2$ at (k_1,k_2) is a tensor product $\Gamma_{k_2}\Gamma_{k_1k_2}^{-1}\Gamma_{k_1}$, hence the section τ of $\delta\Gamma \to K^2$ defines a unitary isomorphism $\Gamma_{k_1}\Gamma_{k_2} \cong \Gamma_{k_1k_2}$, or equivalently a product on \widehat{K} covering the group multiplication on K. Finally, the condition $\delta\tau=1$ is equivalent to associativity of this product.

A pseudo-line bundle (E,s) for the simplicial gerbe (Γ,τ) is the same thing as a splitting of the central extension: Obviously E is trivial since B_0K is just a point; the section s defines a trivialization $\widehat{K} = K \times U(1)$, and $\delta s = t$ means that this is a group homomorphism.

DEFINITION 2.6. A connection on a simplicial gerbe (L,t) over M_{\bullet} is a line bundle connection ∇^L , together with a 2-form $B \in \Omega^2(M_0)$, such that $(\delta \nabla^L) t = 0$ and

$$\delta B = \frac{1}{2\pi i} \operatorname{curv}(\nabla^L).$$

Given a pseudo-line bundle $\mathcal{L}=(E,s)$, we say that ∇^E is a pseudo-line bundle connection if it has the property $((\delta\nabla^E)^{-1}\nabla^L)s=0$.

Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the δ -cohomology of the double complex $\Omega^k(M_n)$ vanishes in bidegrees (1,2) and (2,1). In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion $\pi: X \to M$ the sequence

$$(2.1) 0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \cdots$$

is exact, so the δ -cohomology vanishes in *all* degrees.

Thus, every bundle gerbe $\mathcal{G}=(X,L,t)$ over a manifold M (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the 3-curvature $\eta\in\Omega^3(M)$ of the bundle gerbe connection by $\pi^*\eta=\mathrm{d}\,B\in\ker\delta$. It can be shown that its cohomology class is the image of the Dixmier-Douady class $[\mathcal{G}]$ under the map $H^3(M,\mathbf{Z})\to H^3(M,\mathbf{R})$. Similarly, if \mathcal{G} admits a pseudo-line bundle $\mathcal{L}=(E,s)$, one can always choose a pseudo-line bundle connection ∇^E . The difference $\frac{1}{2\pi i}\operatorname{curv}(\nabla^E)-B$ is δ -closed and one defines the error 2-form of this connection by

$$\pi^*\omega = \frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B.$$

It is clear from the definition that $d\omega + \eta = 0$.

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

2.4 Equivariant bundle gerbes

Suppose G is a Lie group acting on X and on M, and that $\pi: X \to M$ is a G-equivariant surjective submersion. Then G acts on all fiber products $X^{[p]}$. We will say that a bundle gerbe $\mathcal{G} = (X, L, t)$ is G-equivariant, if L is a G-equivariant line bundle and t is a G-invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction $X_G = EG \times_G X \to M_G = EG \times_G M$, hence has an equivariant Dixmier-Douady class in $H^3(M_G, \mathbf{Z}) = H^3_G(M, \mathbf{Z})$. Similarly, we say that a pseudo-line bundle (E, s) for (X, L, t) is equivariant, provided E carries a G-action and E is an invariant section.

¹) We have not discussed bundle gerbes over infinite-dimensional spaces such as M_G . Recall however [4] that the classifying bundle $EG \to BG$ may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.